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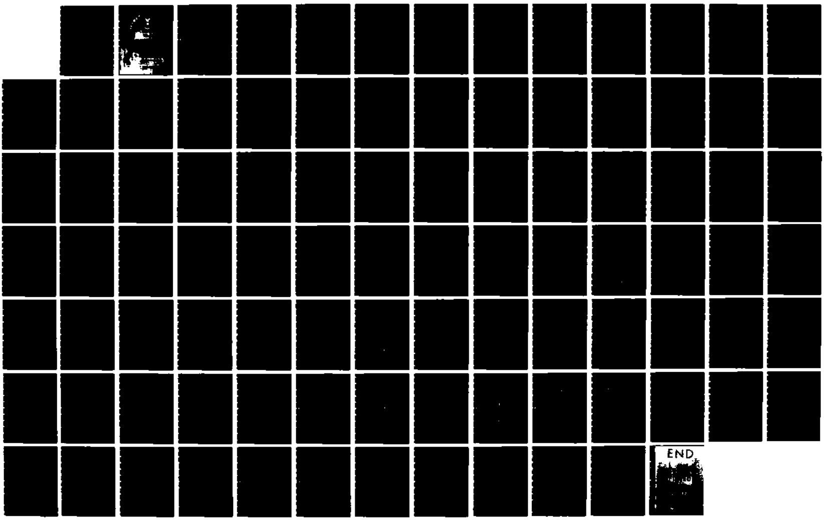

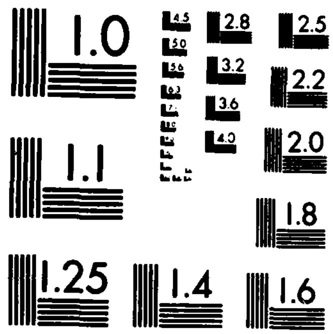

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A VARIATIONAL METHOD FOR CALCULATING THE NATURAL FREQUENCIES AND MODE SHAPES OF A CANTILEVERED OPEN CYLINDRICAL SHELL

THESIS

Jeffrey V. Kouri
Captain, USAF

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A VARIATIONAL METHOD FOR CALCULATING THE NATURAL FREQUENCIES AND MODE SHAPES OF A CANTILEVERED OPEN CYLINDRICAL SHELL

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

A1

Jeffrey V. Kouri

Captain, USAF

Graduate Aeronautical Engineering

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Approved for public release; distribution unlimited

Preface

I would like to take this opportunity to express my sincere gratitude to my thesis advisor Dr. Peter J. Torvik. The knowledge I have gained through him during the course of this study far exceeds that which has actually been recorded on the pages of this thesis.

Most of all I want to thank my wife, Kris, for not only her help in putting together this study, but especially for her patience, encouragement and understanding during the entire program.

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List of Symbols

u_i^*	Prescribed displacements
u_i	Generalized displacements
W	Elastic strain energy density
F_i	Body forces
T_i^*	Prescribed surface tractions
V	Volume of system
S	Surface of system (S_u+S_σ)
S_u	Portion of S where displacements are prescribed
S_σ	Portion of S where tractions are prescribed
σ_{ij}	Stress components
ε_{ij}	Eulerian strains
λ_{ij}, X_n, Y_i	Lagrange multipliers
η_j	Surface direction cosines
M_{nn}	Shell moment resultants. Positive about the positive s axis
x,θ,z	Shell coordinate directions
N_{ni}	Shell force resultants
Q_{nz}	Kirchoff equivalent shear force ($N_{nz}+\partial M_{ns}/\partial S$)
x,v,w	Shell displacements in the X, θ and z directions respectively

K_{nn}	Shell curvatures
E	Young's modulus
h	Shell thickness
ν	Poisson's ratio
ρ	Shell mass density
β	Rotations about the negative S axis
β^*	Presumed rotations
s	Tangential coordinate. Positive counter-clockwise when viewed from the +z direction.
∇^2	Laplace's operator
$\hat{n}$	Surface unit normal
t	Time
$A_i B_i C_i$	Undetermined coefficients in the u,v and w displacement solutions
Ψ_i	A_i/C_i
Φ_i	B_i/C_i
Γ_i	C_i/C_1
λ_i	Roots to Eq (32)
α	See Fig. 3. One-half of angle subtended by shell
ω	Frequency (rads/sec)
E_n	Solution function undetermined coefficients
Q	System matrix

Abstract

This report develops a variational technique for the analysis of the vibration characteristics of an open cylindrical cantilevered shell. The technique is developed by modifying Reissner's principle, which normally applies to static problems, through the use of Hamilton's principle so that it applies to dynamic problems. The variational technique is first derived in general for an elastic system, and then specifically tailored to an open cylindrical cantilevered shell. The technique is implemented by first finding a general solution which satisfies the equations of motion for a cylindrical shell. A method is then formulated to use this general solution to construct a set of trial solution functions. With the variational method, the coefficients to this trial *solution function are then* calculated so that the function not only satisfies the equations of motion, but also the boundary conditions around the four edges of the shell. A computer method was developed to perform the necessary calculations to implement the variational procedure, but preliminary results have shown that numerical problems must be eliminated before accurate results can be expected.

Experimental data for an open cylindrical cantilevered shell was also collected on a modal analyzer. The results are presented and discussed.

I. Introduction

Background

The vibration characteristics of a cantilevered open shallow cylindrical shell of rectangular planform are quite different than those of a similarly dimensioned flat cantilevered plate. The determination of these differences is of significant technical importance in areas such as in the analysis of turbomachinery blades where accurate natural frequencies and mode shapes are desirable. Unfortunately, tabulated vibration data is not as readily available for cantilevered shells as it is for plates and beams. In fact, in Vibration of Shells by Leissa (Ref 1), one of the most comprehensive collections of data in this area, there is no numerical data given for a cantilevered shell as described above. This lack of data is a result of the equations of motion becoming more complex in going from a plate to a shell. This increase in complexity allows exact solutions to but a few special cases of boundary conditions. For open cylindrical shells some of these special cases include those shells which have shear diaphragms supporting at least two opposite edges. Basically, the shear diaphragm is the analog to "simply supported" in linear beam and plate theory. A shear diaphragm can resist all but the translation normal to the edge and the bending moment along the edge. Therefore, the shear diaphragm does not represent the necessary boundary conditions needed to model a cantilevered shell. As a result, even though general solutions to the equations of motion do exist, most practical boundary conditions require the use of approximate techniques such as the Galerkin or the Rayleigh-Ritz methods. Unfortunately, these

methods do require trial functions which must approximately satisfy at least the geometric boundary conditions. This requirement, once again restricts the class of solvable problems to certain types of boundary conditions. To overcome this restriction, numerical methods such as finite elements and finite differences are commonly used. In an effort to develop a continuum approach, this study will investigate a variational method derived from extending Reissner's principle via Hamilton's principle. The method generates a solution satisfying the boundary conditions by a superposition of a set of functions, each of which satisfies the governing equations of motion, but not necessarily the boundary conditions. This method has been successfully used to determine the natural frequencies of a partially clamped circular plate (Ref 2).

Objective and Approach

The objective of this study is to determine if it is possible and practical to apply a variational procedure to calculate the free vibration characteristics of shells with mixed boundary conditions. In particular, the procedure will be used on a cantilevered open cylindrical shell.

The method will be developed using thin elastic shell theory specialized to the case of cylindrical shells, and then applied to an open cantilevered cylindrical shell. A procedure will then be devised to calculate the coefficients necessary to superimpose trial functions to construct an approximate solution. A computer program will be developed that implements the variational procedure. Experimental data collected on a modal analyzer will also be presented and discussed.

II. Extension of Reissner's Principle

Reissner's Principle

Reissner's principle belongs to a class of variational techniques known as stationary principles. Unlike the principles of minimum total potential energy and minimum total complimentary energy, which ensure the extremum found is a minimum, Reissner's principle ensures only that a stationary value is found. The exact nature of the extremum is not known. Reissner's principle is a generalization of the principle of stationary potential energy through the use of Lagrange multipliers.

The principle of stationary potential energy is derived from the principle of virtual work by assuming that the applied loads are conservative and do not vary in magnitude or direction during virtual displacements (Ref 8:67). This principle can be stated as

$$\delta I = 0 \tag{1}$$

where

$$I = \int_V \left(W(u_i) - F_i u_i \right) dV - \int_{S_\sigma} T_i^* u_i ds \tag{2}$$

and

$W(u_i)$ = Strain energy density of system in terms of displacements and their derivatives

u_i = Displacements

F_i = Body forces per unit volume

T_i^* = Prescribed surface tractions

V = Total volume of system

S_σ = Portion of boundary where tractions are prescribed

Reissner generalized this functional to allow simultaneous artibrary variations in displacement, stress, and strain by adding constraints, each multiplied by Lagrange multipliers to give

$$I = \int_V \left[W(\varepsilon_{ij}) - F_i u_i\right] dV - \int_{S_\sigma} T_i^* u_i ds$$

$$- \int_V \lambda_{ij} \left[\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})\right] dV - \int_{S_u} Y_i (u_i - u_i^*) ds \tag{3}$$

The first constraints, (with Lagrange multiplier, λ_{ij}), are seen to ensure satisfaction of the strain displacement equations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{in } V \tag{4}$$

(where commas denote partial differentiation), and the second constraint, (with multiplier, Y_i), ensures

$$u_i = u_i^* \qquad \text{on } S_u \tag{5}$$

where S_u is the portion of the boundary where displacements are prescribed, and u_i^* are the prescribed displacements on S_u.

In taking the first variation of Eq (3), not only are the displacements allowed to vary, but so too are the strains and Lagrange multipliers. It can easily be shown (Ref 9:5) that the vanishing of

this variation requires

$$\int_V \left[(\partial W/\partial \varepsilon_{ij}) - \lambda_{ij}\right] \delta\varepsilon_{ij}\, dV - \int_V (\lambda_{ij,j} + F_i)\delta u_i\, dV$$

$$- \int_V \left[\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})\right] \delta\lambda_{ij} dV - \int_{S_\sigma} (T_i^* - \eta_j\lambda_{ij})\delta u_i ds$$

$$- \int_{S_u} (u_i - u_i^*)\delta Y_i ds - \int_S (Y_i - \eta_j\lambda_{ij})\delta u_i ds = 0 \qquad (6)$$

where the η_j are the direction cosines of the surface, and $S = S_\sigma + S_u$. If the λ_{ij} are taken to be components of stress, and the Y_i components of traction, the first three volume integrals and the last surface integral represent the stress strain laws, the equations of equilibrium, the strain displacement equations and Cauchy's law, respectively. Thus, any displacement field which satisfies these equations ensures that all but the last two surface integrals vanish. The equation left to be satisfied has then been reduced to

$$\int_{S_\sigma} (\eta_j\sigma_{ij} - T_i^*)\delta u_i ds - \int_{S_u} (u_i - u_i^*)\delta T_i ds = 0 \qquad (7)$$

These remaining stress and displacement boundary integrals become very important tools in determining functions which are solutions to a given problem.

Dynamic Form of Reissner's Principle

Reissner's principle, as discussed above, can be applied only to static systems, but can easily be extended to include dynamics with the use of Hamilton's principle. To this end, Eq (3) can be turned into a Lagrangian function, J, by subtracting the system kinetic energy, K, represented by

$$K = \int_V \tfrac{1}{2}\, \rho \dot{u}_i \dot{u}_i \, dV \tag{8}$$

where ρ is the mass density and the dots represent derivatives with respect to time. As per Hamilton's principle, (Ref 10), the resulting Lagrangian is integrated over time from t_1 to t_2 to give

$$J = \int_{t_1}^{t_2} \left\{ \int_V \left[W(\varepsilon_{ij}) - F_i u_i - \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i \right] dV - \int_{s_\sigma} T_i^* u_i \, ds \right.$$

$$\left. - \int_V \lambda_{ij} \left[\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i}) \right] dV - \int_{s_u} Y_i (u_i - u_i^*) \, ds \right\} dt = 0 \tag{9}$$

This time dependent version of Reissner's principle will become the basis for further development in this study, and upon setting the first variation to zero, will be shown to produce the equations of motion and boundary conditions of the system being studied. The boundary conditions will take on the same form as those given in Eq (7).

III. Specialization of the Variational Principle to Cylindrical Shells

The application of the previously described variational procedure to a cylindrical shell requires a slight modification of Eq (9) to ensure the necessary form of each of its terms. Before doing this however, Eq (9) can be simplified somewhat by applying the strain displacement equations directly to the strain energy expression by expressing W only in terms of displacements. In doing so, the corresponding Lagrange multiplier constraint term on strain and displacements can be eliminated from the equation. If this is not done, an additional Lagrange multiplier is needed when Eq (9) is applied to a shell. This is because for a shell, W not only becomes a function of mid-plane strains, but also of curvature. Since curvature must also be allowed virtual changes, this third set of Lagrange multipliers is needed to ensure satisfaction of the shell curvature displacement relations. Whether Eq (9) is employed with W expressed in terms of strains and curvatures in conjunction with their respective constraint equations, or whether W is put in terms of only displacements without the constraints is a matter of personal preference. In the preparation of this study, both methods were employed with identical results. The latter will be described herein.

In this light, for a shell with no body forces Eq (9) can be expressed as

$$J = \int_{t_1}^{t_2} \left\{ \int_V \left(W(u_i) - \tfrac{1}{2} \rho \dot{u}_i \dot{u}_i \right) dV - \int_{S_\sigma} (-M_{nn}{}^* \beta + Q_{ni}{}^* u_i) ds \right.$$

$$\left. - \int_{S_u} \left(X_n(\beta - \beta^*) + Y_i(u_i - u_i{}^*) \right) ds \right\} dt \qquad (10)$$

where ($i = 1,2,3$ and $n = x,\theta$)

$M_{nn}{}^*$ = Prescribed moments on S_σ, positive about the positive s axis

$Q_{ni}{}^*$ = Prescribed stress resultants on S_σ defined by: $Q_{ni} = N_{ni} + (\partial M_{ns}/\partial s)$ when $i = 3$, (the z direction) and by $Q_{ni} = N_{ni}$ when $i \neq 3$ (the x and θ direction)

β = Rotations about the negative s axis

β^* = Prescribed rotations on S_u

X_n, Y_i = Lagrange multipliers

(The subscript i follows the rules of the summation convention, but the subscript n does not.) After taking the first variation of Eq (10), the Lagrange multipliers will form complimentary virtual work terms similar to those in Eq (6). It follows, the Y_i are tractions and the X_n is the negative of the moment about the s axis.

The application of Eq (10) to cylindrical shells now requires the representation of the elastic strain energy density of the shell as a

function of the displacements of u, v, w, and their derivatives. Towards this end, a shell coordinate system is defined as shown in Fig. 1. For this system the thin elastic shell theory strain displacement equations for mid-surface strains reduce to (Ref 3)

$$
\begin{aligned}
\varepsilon_x &= u,_x \\
\varepsilon_\theta &= \frac{1}{a}(v,_\theta + w) \\
\gamma_{x\theta} &= \frac{1}{a}u,_\theta + v,_x
\end{aligned}
\tag{11}
$$

while the expressions for curvature become

$$
\begin{aligned}
K_x &= -w,_{xx} \\
K_\theta &= (\frac{1}{a})2(-w,_{\theta\theta} + v,_\theta) \\
K_{x\theta} &= \frac{1}{a}(-w,_{x\theta} + \tfrac{1}{2}v,_x)
\end{aligned}
\tag{12}
$$

The expression for strain energy density per unit area for an isotropic cylindrical shell can be written as (Ref 4)

$$
\begin{aligned}
W(\varepsilon_{ij}, K_{ij}) = {} & \tfrac{1}{2}D_1 \left[(\varepsilon_x + \varepsilon_\theta)^2 - 2(1-\nu)(\varepsilon_x\varepsilon_\theta - \gamma_{x\theta}{}^2/4)\right] \\
& + \tfrac{1}{2}D_2 \left[(K_x + K_\theta)^2 - 2(1-\nu)(K_x K_\theta - K_{x\theta}{}^2)\right]
\end{aligned}
\tag{13a}
$$

where

$$D_1 = (Eh)/(1-\nu^2) \tag{13b}$$

$$D_2 = (Eh^3)/12(1-\nu^2) \tag{13c}$$

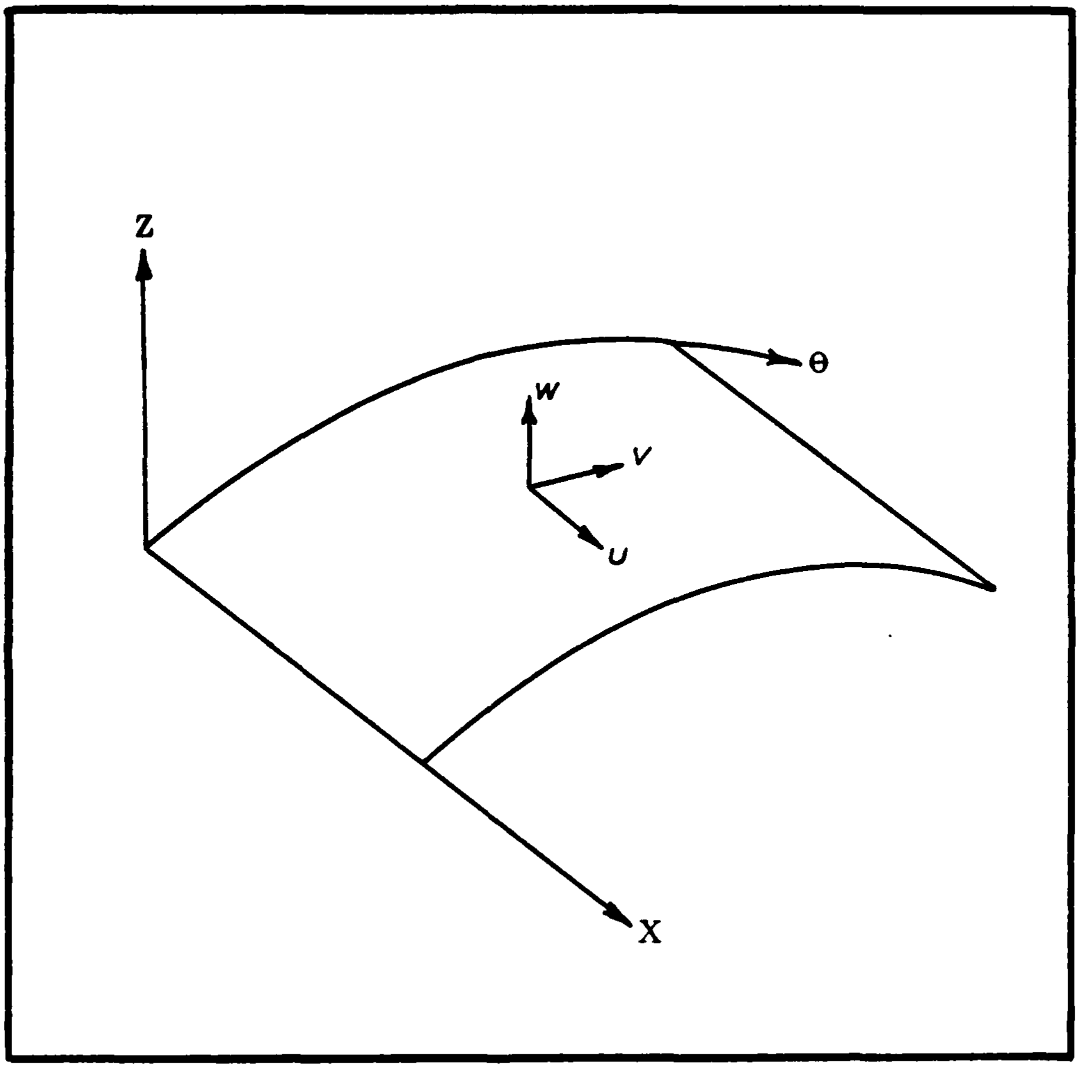

Figure 1. Shell Coordinate System

Substituting Eqs (11), and (12) into Eq (13a) produces W in terms of u, v, and w. Substituting this resultant expression into Eq (10) and setting the first variation of the functional to zero produces

$$\int_{t_1}^{t_2} \Bigg\{ \tfrac{1}{2} D_1 \int_A \Bigg(2\left[u,_x + \frac{1}{a}\nu(v,_\theta + w)\right]\delta u,_x + 2\left[\left(\frac{1}{a}\right)^2 (v,_\theta + w) + \frac{1}{a}\nu u,_x\right]\delta v,_\theta$$

$$+ \frac{2}{a}\left[\frac{1}{a}(v,_\theta + w) + \nu u,_x\right]\delta w + (1-\nu)\left(\frac{1}{a}u,_\theta + v,_x\right)\delta v,_x$$

$$+ \left[\frac{1}{a}(1-\nu)\left(\frac{1}{a}u,_\theta - v,_x\right)\right]\delta u,_\theta \Bigg) dA$$

$$+ D_2 \int_A \Bigg(\left[w,_{xx} + \nu\left(\frac{1}{a}\right)^2 (w,_{\theta\theta} - v,_\theta)\right]\delta w,_{xx}$$

$$+ \left(\frac{1}{a}\right)^4 \left[(w,_{\theta\theta} - v,_\theta + a^2 \nu w,_{xx}\right]\delta w,_{\theta\theta}$$

$$- 2\left(\frac{1}{a}\right)^2 (1-\nu)\left[(-w,_{x\theta} + \tfrac{1}{2}v,_x)\right]\delta w,_{x\theta}$$

$$- \left(\frac{1}{a}\right)^4 \left[w,_{\theta\theta} - v,_\theta + a^2 \nu w,_{xx}\right]\delta v,_\theta$$

$$+ (1-\nu)\left(\frac{1}{a}\right)^2 \left[-w,_{x\theta} + \tfrac{1}{2}v,_x\right]\delta v,_x \Bigg) dA$$

$$- \int_A \rho \dot{u}_i \delta \dot{u}_i dA - \int_{s_\sigma} (-M_{nn}{}^* \delta\beta + Q_{ni}{}^* \delta u_i) ds - \int_{s_\sigma} (X_n \delta\beta + Y_i \delta u_i) ds$$

$$- \int_{s_u} (\beta-\beta^*)\delta X_n ds - \int_{s_u} (u_i-u_i^*)\delta Y_i \Bigg\} dt = 0 \qquad (14)$$

At this point, it is also convenient to define the force and moment resultants for thin shells (Ref 3) as

$$\begin{aligned}
N_{xx} &= D_1(\varepsilon_x+\nu\varepsilon_\theta) & M_{xx} &= D_2(K_x+\nu K_\theta) \\
N_{\theta x} &= N_{x\theta} = D_1/2(1-\nu)\gamma_{x\theta} & M_{\theta x} &= M_{x\theta} = D_2(1-\nu)K_{x\theta} \\
N_{\theta\theta} &= D_1(\varepsilon_\theta+\nu\varepsilon_x) & M_{\theta\theta} &= D_2(K_\theta+\nu K_x) \\
N_{xz} &= \frac{1}{a}(M_{xx,x}+\frac{1}{a}M_{\theta x,\theta}) & N_{\theta z} &= \frac{1}{a}(\frac{1}{a}M_{\theta\theta,\theta}+M_{x\theta,x})
\end{aligned} \qquad (15)$$

These are shown pictorially in Fig. 2.

If Eqs (11) and (12) are substituted into Eqs (15), and those results placed into Eqs (14), the functional can be written as

$$\int_{t_1}^{t_2} \Bigg\{ \int_A (N_{xx}\delta u,_x + \frac{1}{a}N_{\theta\theta}\delta v,_\theta + \frac{1}{a}N_{\theta\theta}\delta w$$

$$+ N_{x\theta}\delta v,_x + \frac{1}{a}N_{x\theta}\delta u,_\theta)dA$$

$$+ \int_A (-M_{xx}\delta w,_{xx} - (\frac{1}{a})^2 M_{\theta\theta}\delta w,_{\theta\theta} - \frac{2}{a}M_{x\theta}\delta w,_{x\theta})dA$$

$$+ \int_A ((\frac{1}{a})^2 M_{\theta\theta}\delta v,_\theta + \frac{1}{a}M_{x\theta}\delta v,_x)dA - \int_A \rho\dot{u}_i\delta\dot{u}_i dA$$

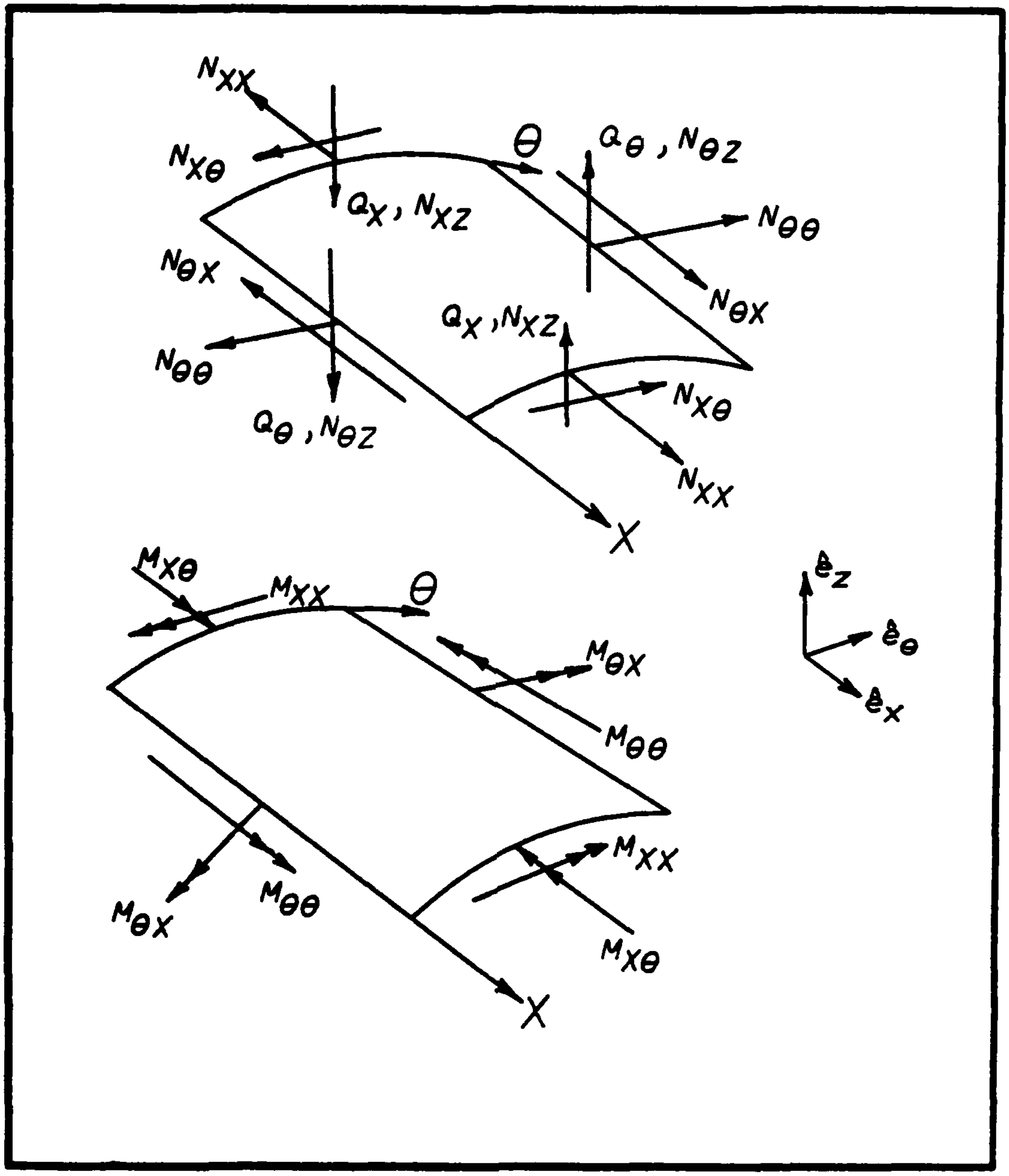

Figure 2. Shell Force and Moment Resultants

$$- \int_{s_\sigma} (-M_{nn}{}^* \delta\beta + Q_{ni}{}^* \delta u_i) ds - \int_{s_u} (X_n \delta\beta + Y_i \delta u_i) ds$$

$$\left. - \int_{s_u} (\beta - \beta^*) \delta X_n ds - \int_{s_u} (u_i - u_i{}^*) \delta Y_i ds \right\} dt \tag{16}$$

To integrate the first area integral in Eq (16), call it I_1, the following vectors are defined:

$$\phi_1 = N_{xx} \delta u \, \hat{e}_x + N_{x\theta} \delta u \, \hat{e}_\theta \tag{17a}$$

$$\phi_2 = N_{x\theta} \delta v \, \hat{e}_x + N_{\theta\theta} \delta v \, \hat{e}_\theta \tag{17b}$$

With these

$$I_1 = \int_A \nabla \cdot \phi_1 dA + \int_A \nabla \cdot \phi_2 dA + \int_A \frac{1}{a} N_{\theta\theta} \delta w dA$$

$$- \int_A (N_{xx,x} + \frac{1}{a} N_{x\theta,\theta}) \delta u dA - \int_A (N_{x\theta,x} + \frac{1}{a} N_{\theta\theta,\theta}) \delta v dA \tag{18}$$

Application of the divergence theorem gives

$$I_1 = \int_A \left(\frac{1}{a} N_{\theta\theta} \delta w - (N_{xx,x} + \frac{1}{a} N_{x\theta,\theta}) \delta u - (N_{x\theta,x} + \frac{1}{a} N_{\theta\theta,\theta}) \delta v \right) dA$$

$$+ \int_s \left((n_x N_{xx} + n_\theta N_{x\theta}) \delta u + (n_x N_{x\theta} + n_\theta N_{\theta\theta}) \delta v \right) ds \tag{19}$$

where η_γ are defined as the direction cosines between the normal and the γ direction. To integrate the second integral of Eq (16), I_2, four scalars are defined as

$$\nabla\Psi_x = M_{xx}\,\hat{e}_x + M_{x\theta}\,\hat{e}_\theta$$

$$\nabla\Psi_\theta = M_{x\theta}\,\hat{e}_x + M_{\theta\theta}\,\hat{e}_\theta$$

$$\phi_x = \delta w,_x \tag{20}$$

$$\phi_\theta = (1/a)\delta w,_\theta$$

With Eqs (20), I_2 can be written (summing on γ, where γ = x or θ)

$$I_2 = -\int_A (\nabla\phi_\gamma \cdot \nabla\Psi_\gamma)\,dA \tag{21}$$

Then, with the use of Green's identity,

$$I_2 = \int_A \phi_\gamma \nabla^2 \Psi_\gamma\,dA - \int_s \phi_\gamma (\partial\Psi_\gamma/\partial\eta)\,ds \tag{22}$$

Equation (22) can be put in a more convenient form by defining

$$\nabla\chi = M_{\alpha\beta,\alpha}(\hat{e}\beta) \tag{23}$$

Now, Eq (22) becomes

$$I_2 = \int_A \nabla(\delta w)\cdot\nabla\chi\ dA - \int_s \phi_\gamma (\partial\Psi_\gamma/\partial\eta)\,ds \tag{24}$$

If the divergence theorem is applied to Eq (24), the result is

$$I_2 = -\int_A (M_{\alpha\beta,\alpha\beta})\delta w\ dA + \int_s \eta_\beta (M_{\alpha\beta,\alpha})\delta w\ ds$$

$$-\int_s \delta w,_x (\partial\Psi_x/\partial\eta)ds - \int_s \frac{1}{a}\delta w,_\theta (\partial\Psi_\theta/\partial\eta)ds \tag{25}$$

or

$$I_2 = -\int_A (M_{\alpha\beta,\alpha\beta})\delta w\ dA + \int_s \eta_\beta (M_{\alpha\beta,\alpha})\delta w\ ds$$

$$-\int_s \delta w,_x (\hat{n}\cdot\nabla\Psi_x)ds - \int_s \frac{1}{a}\delta w,_\theta (\hat{n}\cdot\nabla\Psi_\theta)ds \tag{26}$$

where $\hat{n}$ is the unit normal of the surface. Finally, after more integration by parts, some manipulation, and setting $\delta u_i(t_1) = \delta u_i(t_2) = 0$, Eq (16) can be written as

$$\int_{t_1}^{t_2} \Big\{ \int_A \Big[(\frac{1}{a}N_{\theta\theta} - M_{xx,xx} - \frac{2}{a}M_{x\theta,x\theta} - (\frac{1}{a})^2 M_{\theta\theta,\theta\theta} + \rho\ddot{w})\delta w$$

$$- (N_{x\theta,x} + \frac{1}{a}N_{\theta\theta,\theta} + \frac{1}{a}M_{x\theta,x} + (\frac{1}{a})^2 M_{\theta\theta,\theta} - \rho\ddot{v})\delta v$$

$$- (N_{xx,x} + \frac{1}{a}N_{x\theta,\theta} - \rho\ddot{u})\delta u \Big] dA$$

$$- \int_s \left(\left[\hat{n} \cdot (M_{xx}\hat{e}_x + M_{x\theta}\hat{e}_\theta) \right] \delta w,_x + \frac{1}{a} \left[\hat{n} \cdot (M_{x\theta}\hat{e}_x + M_{\theta\theta}\hat{e}_\theta) \right] \delta w,_\theta \right) ds$$

$$+ \int_s \left[(n_x N_{xx} + n_\theta N_{x\theta}) \delta u + (n_x N_{x\theta} + n_\theta N_{\theta\theta}) \delta v \right] ds$$

$$+ \int_s \left[n_x M_{xx,x} + n_\theta M_{x\theta,x} + \frac{1}{a} (n_x M_{x\theta,\theta} + n_\theta M_{\theta\theta,\theta}) \delta w \right] ds$$

$$+ \frac{1}{a} \int_s \left[\hat{n} \cdot (M_{x\theta}\hat{e}_x + M_{\theta\theta}\hat{e}_\theta) \right] \delta v \, ds - \int_{s_\sigma} (-M_{nn}{}^* \delta\beta + Q_{ni}{}^* \delta u_i) ds$$

$$- \int_{s_u} (X_n \delta\beta + Y_i \delta u_i) ds - \int_{s_u} (\beta - \beta^*) \delta X_n ds$$

$$- \int_{s_u} (u_i - u_i{}^*) \delta Y_i ds \Bigg\} dt = 0 \qquad (27)$$

The three terms in the area integral of Eq (27) can be recognized as the three time-dependent equations of equilibrium for a cylindrical shell (Ref 3:200). The line integrals, which evolved from integration by parts, represent the boundary conditions which need to be satisfied on the surface of the shell, and will become more readily recognizable when applied to a specific shell shape. Finding a set of displacement

functions which satisfies this equation ensures the first variation of the function vanishes, and hence that an extremum has been found.

IV. "Exact" Solutions for a Cylindrical Shell

The Dynamic Donnell Equations

Finding a general solution which satisfies the equations of motion given in Eq (27) can be facilitated by following a procedure used by Kraus (Ref 3:302-304). The first step is to apply the Donnell assumptions, which Donnell proposed in 1933 to simplify the cylindrical shell equations (Ref 6). In his first assumption, Donnell argued that the transverse shearing force makes a negligible contribution to the equilibrium of forces in the circumferential direction (Ref 1:200). In the second group of terms in the area integral of Eq (27), the terms which make up this force are

$$N_{\theta z} = \frac{1}{a}(M_{x\theta,x} + \frac{1}{a}M_{\theta\theta,\theta}) \approx 0$$

and can thus be eliminated. Substitution of Eqs (11), (12) and (15) into the equations of motion from Eq (27) (given in the first area integral after this first Donnell assumption has been made), allows the equations to be represented in terms of u, v, and w. Through a series of operations and substitutions, the following three new equations can be derived

$$\nabla^4(u) + (\frac{\nu}{a})w_{,xxx} - (\frac{1}{a})^3 w_{,x\theta\theta}$$

$$= -\frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left[\frac{1-\nu^2}{E}\rho\ddot{u} - \frac{3-\nu}{2}\nabla^2(u) - (\frac{\nu}{a})w_{,x}\right] \qquad (28a)$$

$$\nabla^4(v) + (\tfrac{1}{a})^2(2+\nu)w_{,xx\theta} + (\tfrac{1}{a})^4 w_{,\theta\theta\theta}$$

$$= -\frac{2(1+\nu)}{E}\rho\,\frac{\partial^2}{\partial t^2}\left[\frac{1-\nu^2}{E}\rho\ddot{v} - \frac{3-\nu}{2}\nabla^2(v) - (\tfrac{1}{a})^2 w_{,\theta}\right] \tag{28b}$$

$$(h^2/12)\nabla^8(w) + (\tfrac{1}{a})^2(1-\nu^2)w_{,xxxx}$$

$$= -\frac{2(1+\nu)}{E}\rho\,\frac{\partial^2}{\partial t^2}\left[\left(\frac{1-\nu^2}{E}\rho\,\frac{\partial^2}{\partial t^2} - \frac{3-\nu}{2}\nabla^2\right)\right.$$

$$\left(\frac{1-\nu^2}{E}\rho\ddot{w} + (\tfrac{1}{a})^2 w + (h^2/12)\nabla^4(w)\right)$$

$$\left. + \tfrac{1}{2}(1-\nu)\nabla^4(w) + (\tfrac{\nu}{a})^2 w_{,xx} + (\tfrac{1}{a})^4 w_{,\theta\theta}\right] \tag{28c}$$

These new equations, which are commonly known as the Donnell equations, are of higher order than the original system of equations, (one eighth order plus two fourth order as compared to one fourth order plus two second order). The advantage to this increase in order is that the displacements have become decoupled. However, any solution that is found to satisfy these new equations should also be checked back in the original system to guard against the introduction of extraneous roots.

Solution Method for Closed Cylindrical Shells

Kraus (Ref 3) outlines a method for determining the natural frequencies for a closed cylindrical shell by first assuming a solution function given by (symmetric modes)

$$u = \sum_{i=1}^{K} A_i \exp(\lambda_i x/l)\cos(n\theta)\cos(\omega t)$$

$$v = \sum_{i=1}^{K} B_i \exp(\lambda_i x/l)\sin(n\theta)\cos(\omega t) \qquad (29)$$

$$w = \sum_{i=1}^{K} C_i \exp(\lambda_i x/l)\cos(n\theta)\cos(\omega t)$$

where A_i, B_i, and C_i are arbitrary constants, and the λ_i are to be determined so that Eqs (29) satisfy the governing equations. Due to the order of the system, the number of terms in the summation, K, can be shown to be eight (Ref 7). If these assumed solutions are substituted into the dynamic Donnell equations, Eqs (28), the result is

$$\frac{A_i}{C_i}\left[\frac{2\Delta^2}{1-\nu} - \frac{(3-\nu)\Delta n^2\gamma}{1-\nu} + n^4\gamma^2\right]$$

$$= -\frac{\lambda_i a}{l}\left[\frac{2\nu\Delta}{1-\nu} + n^2(1+\nu Z)\right] \qquad (30)$$

$$\frac{B_i}{C_i}\left[\frac{2\Delta^2}{1-\nu} - \frac{(3-\nu)\Delta n^2\gamma}{1-\nu} + n^4\gamma^2\right]$$

$$= \left[\frac{2n\Delta}{1-\nu} - n^3(1-(2+\nu)4Z)\right] \qquad (31)$$

$(1-\nu)(1-\nu^2)(\lambda_i a/l)^4$

$$= 2\Delta^3 - \Delta^2 \left[2+(3-\nu)n^2\gamma + (2/\xi)n^4\gamma^2\right]$$

$$+ \Delta\Big[(3-\nu)n^2\gamma - 2n^2(1-\nu^2 Z) + (1-\nu)n^4\gamma^2$$

$$+ (1/\xi)(3-\nu)n^6\gamma^3\Big] - (1/\xi)(1-\nu)n^8\gamma^4 \tag{32}$$

where

$$\Delta = \rho\omega^2 a^2(1-\nu^2)/E \tag{33}$$

$$\xi = 12(a/h)^2 \tag{34}$$

$$Z = (\lambda_i a)^2/(nl)^2$$

and

$$\gamma = (1-Z) \tag{35}$$

At this point, the solution to a closed cylindrical shell becomes fairly straight-forward. There are four boundary conditions to be satisfied at each end. They are (Ref 1:136)

$$N_{xx} = 0 \quad \text{or} \quad u = 0 \tag{36}$$

$$M_{xx} = 0 \quad \text{or} \quad w_{,x} = 0 \tag{37}$$

$$N_{x\theta}+\frac{1}{a}M_{x\theta} = 0 \quad \text{or} \quad v = 0 \tag{38}$$

$$N_{xz}+\frac{1}{a}(M_{x\theta,\theta}) = 0 \quad \text{or} \quad w = 0 \tag{39}$$

If Eqs (29) are substituted into the equations for the eight boundary conditions, the result is eight homogeneous equations in the eight unknowns C_i, (A_i and B_i can be eliminated with Eqs (30) and (31)). The

solution to these eight equations is found by setting the determinant of their coefficients to zero. The determinant is dependent upon the λ_i, which are functions of ω and n through the eighth order Eq (32). Thus, for a given n, and ω is chosen and Eq (32) solved for the eight λ_i. The determinant is then calculated with these values. The procedure is repeated in an iterative manner until an ω is found where the determinant vanishes.

As a point of interest, examination of Eq (32) shows that the eighth order equation in λ_i has only even powers. This leads to a distinct relation between the eight roots; four roots are the negatives of the remaining four.

Unfortunately, the above method cannot be used for open cylindrical shells because four more boundary conditions are introduced on each of the edges of constant θ. They are (Ref 1:157)

$$N_{x\theta} = 0 \quad \text{or} \quad u = 0 \tag{40}$$

$$N_{\theta\theta} = 0 \quad \text{or} \quad v = 0 \tag{41}$$

$$N_{\theta z} + M_{x\theta,x} = 0 \quad \text{or} \quad w = 0 \tag{42}$$

$$M_{\theta\theta} = 0 \quad \text{or} \quad w,_{\theta} = 0 \tag{43}$$

There are now a total of sixteen boundary conditions to be satisfied and only eight undetermined coefficients. Thus, Eqs (29) are not general enough to provide a solution for any one n. For this reason, the variational procedure described earlier will be used in an attempt to find an approximate solution by superposition of functions with several values of n.

V. Variational Method for Open Cylindrical Shells

As discussed in the previous section, the presence of eight additional boundary conditions for the open cylindrical shell prohibits the use of the procedure outlined by Kraus, (Ref 3), to find a solution for the free vibration modes. To overcome this obstacle, the variational procedure developed earlier and expressed by Eq (27) will be used. The approach will be to find a trial solution function which satisfies Eq (27), and hence satisfies or approximates the governing shell equations and all sixteen boundary conditions.

Determination of Surface Integrals

The boundary integrals of Eq (27) can now be expressed in terms of a specific shell shape. Thus far, Eq (27) applies to all cylindrical shells with the only assumption being the thickness, h, is small as compared to the radius, r. Specializing Eq (27) to an open cantilevered shell, such as the one shown in Fig. 3, allows the surface integrals to be written explicitly. Before doing so however, the prescribed force and moment resultants on S_σ, and the prescribed displacements on S_u are all set to zero. This eliminates the terms with the asterisk in the last four integrals of Eq (27). The remaining integrals from Eq (27), if written such that the surface integrations will be done counterclockwise (as viewed from the +z direction) around the four edges with the sign convention of Fig. 2, become

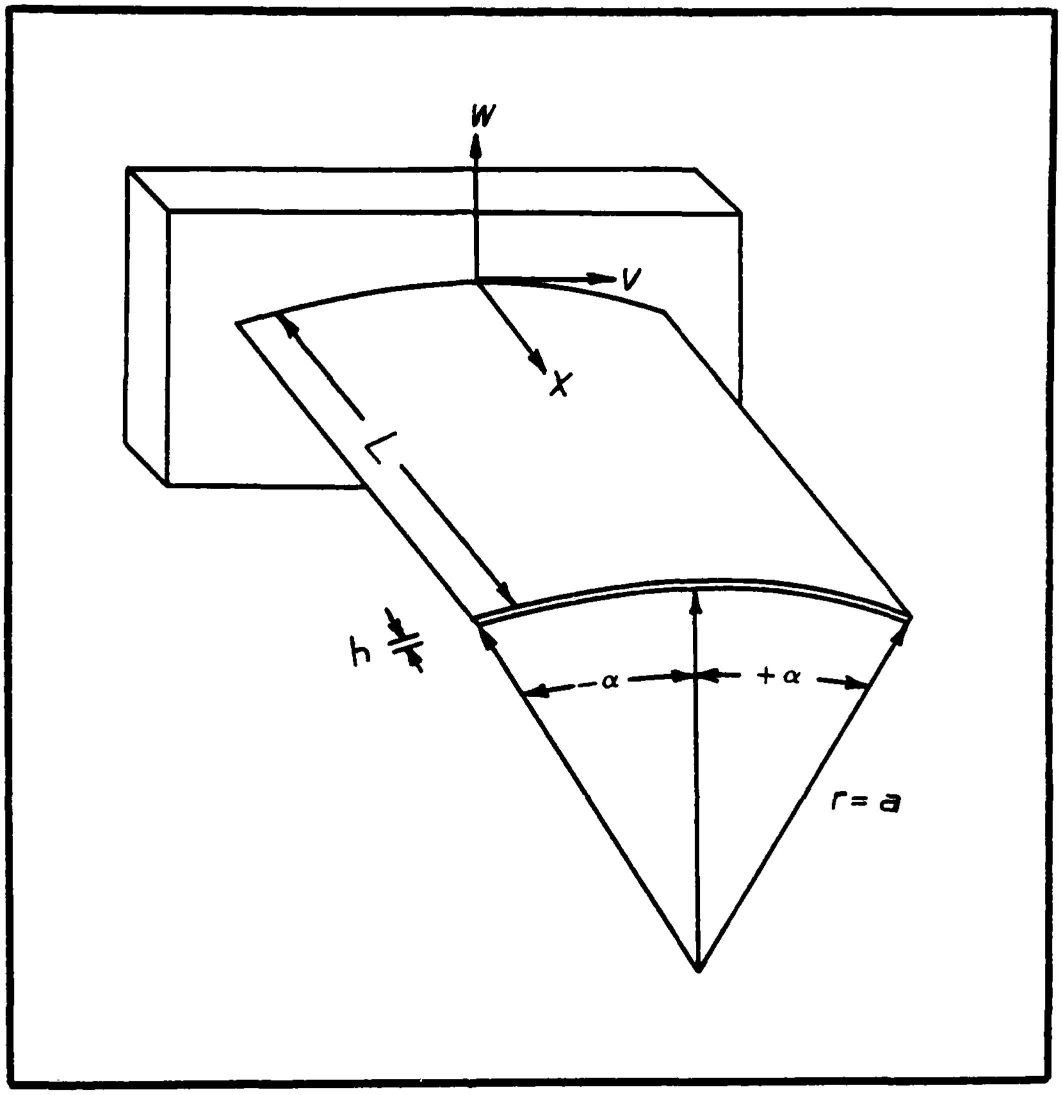

Figure 3. Open Cylindrical Cantilevered Shell

$$\int_{t_1}^{t_2} \left\{ \int_A \left[\left(\frac{1}{a}N_{\theta\theta} - M_{xx,xx} - \frac{2}{a}M_{x\theta,x\theta} - \left(\frac{1}{a}\right)M_{\theta\theta,\theta\theta} - \rho\ddot{w}\right)\delta w \right.\right.$$

$$- \left(N_{x\theta,x} + \frac{1}{a}N_{\theta\theta,\theta} + \frac{1}{a}M_{x\theta,x} + \left(\frac{1}{a}\right)^2 M_{\theta\theta,\theta} + \rho\ddot{v}\right)\delta v$$

$$\left. - \left(N_{xx,x} + \frac{1}{a}N_{x\theta,\theta} + \rho\ddot{u}\right)\delta u \right] dA$$

$$+ \int_{-\alpha}^{\alpha} \left[(N_{xx}\delta u)_{x=1} - (N_{xx}\delta u)_{x=0} \right] a d\theta + \int_{-\alpha}^{\alpha} \left[(N_{x\theta}\delta v)_{x=1} - (N_{x\theta}\delta v)_{x=0} \right] a d\theta$$

$$+ \int_0^1 \left[(N_{x\theta}\delta u)_{\theta=\alpha} - (N_{x\theta}\delta u)_{\theta=-\alpha} \right] dx + \int_0^1 \left[(N_{\theta\theta}\delta v)_{\theta=\alpha} - (N_{\theta\theta}\delta v)_{\theta=-\alpha} \right] dx$$

$$+ \int_{-\alpha}^{\alpha} \left[\left(M_{xx,x} + \frac{2}{a}M_{x\theta,\theta}\right)\delta w \Big|_{x=1} - \left(M_{xx,x} + \frac{2}{a}M_{x\theta,\theta}\right)\delta w \Big|_{x=0} \right] a d\theta$$

$$- \int_0^1 \left[\left(2M_{x\theta,\theta} + \frac{1}{a}M_{\theta\theta,\theta}\right)\delta w \Big|_{\theta=-\alpha} - \left(2M_{x\theta,x} + \frac{1}{a}M_{\theta\theta,\theta}\right)\delta w \Big|_{\theta=\alpha} \right] dx$$

$$+ \int_{-\alpha}^{\alpha} \left[(M_{xx}\delta w_{,x})_{x=0} - (M_{xx}\delta w_{,x})_{x=1} \right] a d\theta$$

$$+ \int_0^1 \left[\frac{1}{a}M_{\theta\theta}\delta(w_{,\theta} - v)_{\theta=-\alpha} - \frac{1}{a}M_{\theta\theta}\delta(w_{,\theta} - v)_{\theta=\alpha} \right] dx$$

$$+\int_{-\alpha}^{\alpha}\left[\frac{1}{a}M_{x\theta}\delta v\Big|_{x=1} - \frac{1}{a}M_{x\theta}\delta v\Big|_{x=0}\right]a d\theta$$

$$-\int_{-\alpha}^{\alpha}\left[M_{xx}\delta w,_{x} - (N_{xz}+\frac{1}{a}M_{x\theta,\theta})\delta w - N_{xx}\delta u - (N_{x\theta}+\frac{1}{a}M_{x\theta})\delta v\right]_{x=0} a d\theta$$

$$-\int_{-\alpha}^{\alpha} w,_{x}\delta M_{xx}\,{}_{x=0}\; a d\theta + \int_{-\alpha}^{\alpha}\Big[u\delta N_{xx} + w\delta(N_{xz}+\frac{1}{a}M_{x\theta,\theta})$$

$$+ v\delta(N_{x\theta}+\frac{1}{a}M_{x\theta})\Big]_{x=0} a d\theta - (4M_{x\theta}\delta w)_{\substack{x=1\\ \theta=\alpha}}\Bigg\}dt = 0 \qquad (44)$$

Several terms in this equation can now be eliminated by inducing the second Donnell assumption (Ref 3). It can be reasoned the changes in the stretching displacement in the circumferential direction, v, have little effect on the curvature and twist of the mid-section. Looking at Eqs (12), this assumption is tantamount to saying derivatives of v are small compared to derivatives of w. This allows the v/a terms to be eliminated in the eighth boundary integral in Eq (44). With both Donnell assumptions, the area integral in Eq (44) represents a new set of equations which again can be transformed into the Donnell equations, Eqs (28), and the boundary terms become the more recognizable shell boundary conditions for an open cylindrical shell given in Eqs (36) through (43).

Determining a Trial Solution Function

In selecting a trial solution function which is to be used to satisfy Eq (44) and hence be a solution to the open shell, it is advantageous to start with one which satisfies as much of Eq (44) term by term as possible. In doing so, those parts of Eq (44) which are satisfied in this manner can be eliminated and the complexity of the calculations reduced. Therefore, if Eqs (29) are used to build a trial function, then the area integral in Eq (44) may be dropped. This is because, as long as Eqs (30), (31) and (32) are enforced, Eqs (29) do in fact satisfy the differential equations of motion.

The flexibility of the variational procedure allows a trial solution to be used which does not have to meet the required boundary conditions term by term. This feature will be employed by using Eqs (29). For example, if Eqs (29) are used to satisfy the necessary boundary condition, $N_{\theta\theta} = 0$, on an edge of constant θ, then the equation that must be satisfied is

$$\left[\exp(\lambda_i x/l)(B_i n/a + C_i/a + \nu A_i \lambda_i/l)\right] \cos(\pm\alpha n) = N_{\theta\theta} = 0 \qquad (45)$$

For $N_{\theta\theta}$ to be zero for any x along a free edge, the term outside the brackets must vanish. This requires that $n=(2m+1)\pi/2\alpha$. If this result is placed into Eqs (29), the displacements u and w are found to vanish along the free edges of constant θ. Since this is totally undesirable for the cantilevered shell, the variational procedure must ensure satisfaction of the boundary conditions on these edges through Eq (44).

At this point, Eqs (30), (31), and (32) can be enforced on Eqs (29), and the result written

$$u = \sum_{i=1}^{8} \Psi_i C_i \exp(\lambda_i x/l)\cos(n\theta)\cos(\omega t)$$

$$v = \sum_{i=1}^{8} \phi_i C_i \exp(\lambda_i x/l)\sin(n\theta)\cos(\omega t) \qquad (46)$$

$$w = \sum_{i=1}^{8} C_i \exp(\lambda_i x/l)\cos(n\theta)\cos(\omega t)$$

where Ψ_i and ϕ_i are determined from Eq (30) and (31) utilizing the λ_i determined from Eq (32).

Thus far, Eqs (46) can be considered each one "term" consisting of eight parts through the constants summed on i from one to eight. If seven boundary conditions are chosen to be satisfied by Eqs (46), seven of the C's can be written in terms of the eighth. Performing this operation also tailors the trial solution functions to be formed as a series of N terms. Each term is of the form of Eqs (46), so it satisfies the seven chosen boundary conditions exactly, term by term. Hence, those corresponding integrals in Eq (44) may be eliminated. The seven boundary conditions that are to be used must be chosen from Eqs (36) through (39). Equations (40) through (43) cannot be used, because as discussed above, Eqs (46) cannot be used to satisfy boundary conditions both along the edges of constant θ and of constant x. A choice must now be made as to which of these seven boundary conditions will be satisfied exactly, and which one is to be left to be approximated through the variational technique. For some problems there are boundary conditions that can be considered to be more crucial than others. For a

cantilevered shell, not requiring that the w displacement at the fixed end be zero would lead to erroneous results in the natural frequencies and mode shapes. However, a v displacement which is left arbitrary at the fixed end represents a shell supported in a channel which restricts motion in all but the surface tangential direction (the v direction). Such a shell could be expected to have natural frequencies and mode shapes very similar to those of a shell with the end totally clamped. Therefore, the boundary condition v=0 at x=0 will be left to be approximately satisfied by the variational principle, rather than exactly.

Equations (46) are now substituted into the necessary seven boundary conditions in Eqs (36) through (39) to set the three remaining displacements to zero at x=0 and the four force and moment resultants to zero at x=l. This produces seven equations in the eight constants C_i. After some manipulation the seven equations can be written

$$\text{If } u(o,\theta) = 0, \quad \text{then} \quad \sum_{i=1}^{8} \Psi_i C_i = 0 \tag{47a}$$

$$\text{If } w(o,\theta) = 0, \quad \text{then} \quad \sum_{i=1}^{8} C_i = 0 \tag{47b}$$

$$\text{If } w,_x(o,\theta) = 0, \quad \text{then} \quad \sum_{i=1}^{8} \lambda_i C_i = 0 \tag{47c}$$

$$\text{If } M_{xx}(l,\theta) = 0, \quad \text{then} \quad \sum_{i=1}^{8} \exp(\lambda_i) C_i \left[-(\lambda_i/l)^2 + \nu(n/a)^2\right] = 0 \tag{47d}$$

$$\text{If } N_{xx}(1,\theta) = 0, \quad \text{then } \sum_{i=1}^{8} \exp(\lambda_i) C_i \left[\Psi_i \lambda_i / l + \frac{1}{a} \nu (n\phi_i + 1) \right] = 0 \tag{47e}$$

$$\text{If } Q_{xz}(1,\theta) = 0, \quad \text{then } \sum_{i=1}^{8} \exp(\lambda_i) C_i \lambda_i \left[(\lambda_i/l)^2 - (2-\nu)(n/a)^2 \right] = 0 \tag{47f}$$

$$\text{If } N_x + \frac{M_x}{a} = 0, \quad \text{then } \sum_{i=1}^{8} \exp(\lambda_i) C_i \Big[D_1 (\phi_i \lambda_i / l - \frac{1}{a} n \Psi_i)$$

$$+ \left(\frac{1}{a}\right)^2 D_2 (1-\nu) \lambda_i n / l \Big] = 0 \tag{47g}$$

Using these seven equations and solving for C_2 through C_8 in terms of C_1 gives

$$C_i = \Gamma_i C_1 \tag{47h}$$

Since the λ_i are complex, Ψ_i, ϕ_i and Γ_i are also complex. For this reason, Eq (47h) cannot be solved by using a standard linear equation solution technique such a Gaussian reduction. Finding the inverse of the solution matrix is one possible way of finding Γ_i.

With this expression Eqs (46) can be written

$$\begin{aligned} u &= C_1 \left[\Gamma_i \Psi_i \exp(\lambda_i x / l) \cos(n\theta) \cos(\omega t) \right] \\ v &= C_1 \left[\Gamma_i \phi_i \exp(\lambda_i x / l) \sin(n\theta) \cos(\omega t) \right] \\ w &= C_1 \left[\Gamma_i \exp(\lambda_i x / l) \cos(n\theta) \cos(\omega t) \right] \end{aligned} \tag{48}$$

The summation on i from this point on will be assumed to be from one to

eight, and the summation sign dropped. Therefore, these three displacement functions will be considered to be one "term" consisting of eight parts. The only unknown constant in Eqs (48) is C_1. Eqs (21) are solutions to the three shell differential equations, and all but one, (v=0 at x=0), of the boundary conditions at the edges of constant x. The constants ϕ_i, Ψ_i, and λ_i are determined for a given ω and n from Eqs (30), (31), and (32). The components of Γ_i are determined from the seven boundary conditions as described above.

The assumed solution functions needed to satisfy Eq (44) can now be defined by creating a series using Eqs (48). Each function consists of N terms given by

$$u = \sum_{n=0}^{N} E_n \ \Gamma_i \Psi_i \exp(\lambda_i x/l) \cos(n\theta) \cos(\omega t)$$

$$v = \sum_{n=0}^{N} E_n \ \Gamma_i \phi_i \exp(\lambda_i x/l) \cos(n\theta) \cos(\omega t) \qquad (49)$$

$$w = \sum_{n=0}^{N} E_n \ \Gamma_i \exp(\lambda_i x/l) \cos(n\theta) \cos(\omega t)$$

Each "term" in these series consists of eight separate parts summed on i. The same constants, E_n, appear in each of the three functions, as did C_1, because the differences between the ratios of u, v, and w are accounted for in the Ψ_i and ϕ_i. Thus, Eqs (49) provide a solution function where each term satisfies exactly the area integral and seven of the boundary integrals of Eq (44). Also, the parameter n, which could have had any value up to this point, has been restricted to

integer values. This choice is acceptable as long as n is chosen so as to include an acceptable class of functions for utilization in the variational procedure. In the assumed solution form, Eq (29), n represents the number of longitudinal nodal lines in the mode shape for a given natural frequency. Thus, n should take on values of 0 through N to provide a "complete" set of functions.

Obtaining a Solution

Equation (44) used in conjunction with Eqs (49) can now be used to solve for the desired vibrational modes for the open cantilevered cylindrical shell. As discussed above, the terms which are satisfied identically by Eqs (49) can be eliminated from Eq (44). This leaves

$$\begin{aligned}\int_{t_1}^{t_2} \Bigg\{ & \int_0^1 \left(N_{x\theta}\delta u\Big|_{\theta=\alpha} - N_{x\theta}\delta u\Big|_{\theta=-\alpha}\right)dx \\ & + \int_0^1 \left(N_{\theta\theta}\delta v\Big|_{\theta=\alpha} - N_{\theta\theta}\delta v\Big|_{\theta=-\alpha}\right)dx \\ & - \int_0^1 \left[\left(2M_{x\theta,x}+\frac{1}{a}M_{\theta\theta,\theta}\right)\delta w\Big|_{\theta=-\alpha} - \left(2M_{x\theta,x}+\frac{1}{a}M_{\theta\theta,\theta}\right)\delta w\Big|_{\theta=\alpha}\right] dx \\ & + \int_0^1 \left[\frac{1}{a}M_{\theta\theta}\delta w,_{\theta}\Big|_{\theta=-\alpha} - \frac{1}{a}M_{\theta\theta}\delta w,_{\theta}\Big|_{\theta=\alpha}\right] dx \\ & + \int_{-\alpha}^{\alpha} v\delta\left(N_{x\theta}+\frac{1}{a}M_{x\theta}\right)_{x=0} a\,d\theta - \left(4M_{x\theta}\delta w\right)_{\substack{x=1\\ \theta=\alpha}} \Bigg\} dt = 0 \qquad (50)\end{aligned}$$

as the equation needed to be satisfied to ensure the functional is at a

stationary value.

The general form of each term in Eq (50) is a force multiplying a virtual displacement, or a displacement multiplying a virtual force. In this light, each term represents a form of virtual work. The functions in Eqs (49) are complex, but the real and imaginary parts each independently satisfy the differential equations. Only the real parts should be used when they are substituted into Eq (50). In this manner each term is a real force multiplying a real displacement and takes the form

$$\int_{t_1}^{t_2} \left(\int_s \mathrm{Re}(F)\ \mathrm{Re}(\delta u)\ \ ds \right) dt$$

resulting in a real work term. Here F is the generalized complex force and u is the generalized complex displacement. If the integration with respect to time is taken to be one period, it can easily be shown that the same results can be achieved by

$$\int_s \tfrac{1}{2}\mathrm{Re}(F\delta\bar{u})\ ds$$

where $\bar{u}$ is the complex conjugate of u. This method will be used when Eqs (49) are substituted into Eq (50). Also in performing the substitution the most general arbitrary variation of a displacement or force resultant is taken to be a variation on the coefficients E_n. For example, substitution of v from Eq (49) into the δv in the first term of Eq (50) would result in

$$\delta v = \delta E_m \phi_j \Gamma_j \cos(n\theta)\cos(\omega t) \quad j=1,2..,8 \tag{51}$$

After substituting in Eqs (49) in this manner and combining terms, Eq (50) can be written

$$\mathrm{Re}\Bigg(\int_0^1 2E_n D_2 \sin(n\alpha)\cos(m\alpha) \Big[\Gamma_i \exp(\lambda_i x/l)(\phi_i\lambda_i/l - \tfrac{n}{a}\Psi_i)\Big]$$

$$\Big[\Gamma_j \Psi_j \phi_j \exp(\lambda_j x/l)\Big] \quad \delta E_m dx$$

$$+ \int_0^1 2E_n D_1 \cos(n\alpha)\sin(m\alpha) \Big[\Gamma_i \exp(\lambda_i x/l)(\tfrac{n}{a}\phi_i + \tfrac{1}{a}\nu\Psi_i\lambda_i/l)\Big]$$

$$\Big[\phi_j \Gamma_j \exp(\lambda_j x/l)\Big] \delta E_m dx$$

$$+ \int_0^1 \tfrac{1}{a} 2nE_n D \sin(n\alpha)\cos(m\alpha) \Big[\Gamma_i \exp(\lambda_i x/l)(2-\nu)(\lambda_i/l)^2 - (\tfrac{n}{a})^2\Big]$$

$$\Big[\Gamma_j \exp(\lambda_j x/l)\Big] \delta E_m dx$$

$$+ \int_0^1 \tfrac{1}{a} 2E_n D \cos(n\alpha)\sin(m\alpha) \Big[\Gamma_i \exp(\lambda_i x/l)((\tfrac{n}{a})^2 - \nu(\lambda_i/l)^2)\Big]$$

$$\Big[m\Gamma_j \exp(\lambda_j x/l)\Big] \delta E_m dx$$

$$+ \int_{-\alpha}^{\alpha} E_n \Big[\Gamma_i \phi_i\Big] \Big[D_2 \Gamma_j (\phi_j \lambda_j/l - \tfrac{1}{a} m\Psi_j)$$

$$+ (\tfrac{1}{a})^2 D(1-\nu)\Gamma_j m\lambda_j/l\Big] \sin(n\theta)\sin(m\theta) \quad a d\theta$$

$$- E_n 4D(1-\nu)\sin(n\alpha)\cos(m\alpha)(\frac{n}{a}\lambda_i/l)(\exp(\lambda_i+\lambda_j)\Gamma_i\Gamma_j)\delta E_m \Bigg) = 0 \qquad (52)$$

Since this expression must be valid for arbitrary values of δE_m, it is convenient to make the choice

$$\delta E_m = 1 \quad \text{for } m=p \quad p=0,1\ldots,M \qquad (53)$$

$$\delta E_m = 0 \quad \text{for } m\neq p \quad p=0,1\ldots,M$$

With this choice Eq (52) becomes a system of M+1 equations each with N+1 terms multiplying the constants E_m. Setting N=M, Equation (52) can be represented by the complex matrix equation

$$Q_{mn} \quad E_n = 0 \qquad n,m = 0,1,2\ldots,N \qquad (54)$$

Each term in the square matrix Q_{mn} is found by solving Eq (52) for the appropriate values of m and n for each coefficient to E_n. It is very important to remember each integral consists of a series of eight terms summed on i multiplying a series of eight terms summed on j. Even though this integration of the product of two series seems laborious, the integrals of Eq (52) reduce to simple summations which are handled very nicely on a digital computer. The solution to the set of equations represented in Eq (54) can be found by setting the determinant of Q to zero. The "eigenvalue" of the system is ω and is implemented through Eq (32). Repeatedly choosing values for ω, performing the necessary calculations to fill Q, and then computing its complex determinant provides a procedure which may be used to find zeroes of det(Q) iteratively,

and thus solve for approximate fundamental frequencies of the shell.

In general, when working with structures which have linear force-displacement relationships, it can be shown through the Betti-Maxwell reciprocal theorem that the system matrix is symmetric. It is expected therefore, that this system derived through the variational procedure should not be an exception (See Appendix A). For complex matrices, the analog to symmetric is the Hermitian form. It is not readily apparent from studying Eq (52) that Q is Hermitian. This topic will be discussed more in the results section.

VI. Analysis of Results

The natural frequencies and mode shapes of a cantilevered open cylindrical shell were determined experimentally with a Hewlett Packard 5423 Structural Dynamics Analyzer. The dynamics analyzer can accurately determine the natural frequencies and mode shapes of structures such as a cantilevered shell, and thus provide an excellent verification of analytic results.

Experimental Procedures

The HP Structural Dynamics Analyzer is a two channel fast Fourier series spectrum analyzer specifically designed for the purpose of calculating mode shapes and natural frequencies of structures. The analyzer performs this function utilizing two input signals generated from impact testing. The signals are sent to the analyzer in the form of (1) a structure input signal from a force transducer providing the impact time history and (2) a structural output, or response time history, from an accelerometer measuring the structure response to the impact. A sample plot for each of these signals is given in Fig. 4. This figure is representative of the response at one point due to an impact at another. The input signal time history is clearly depicted as a force impulse, while the output shows the resultant vibration response. A transformed ratio of this output signal to the input signal provides a transfer function for the respective two points. Figure 5 represents the transfer function of the measurements given in Fig. 4. The spikes, or commonly called poles, in the plot at various frequencies are a result of the transfer function's denominator approaching zero, causing a

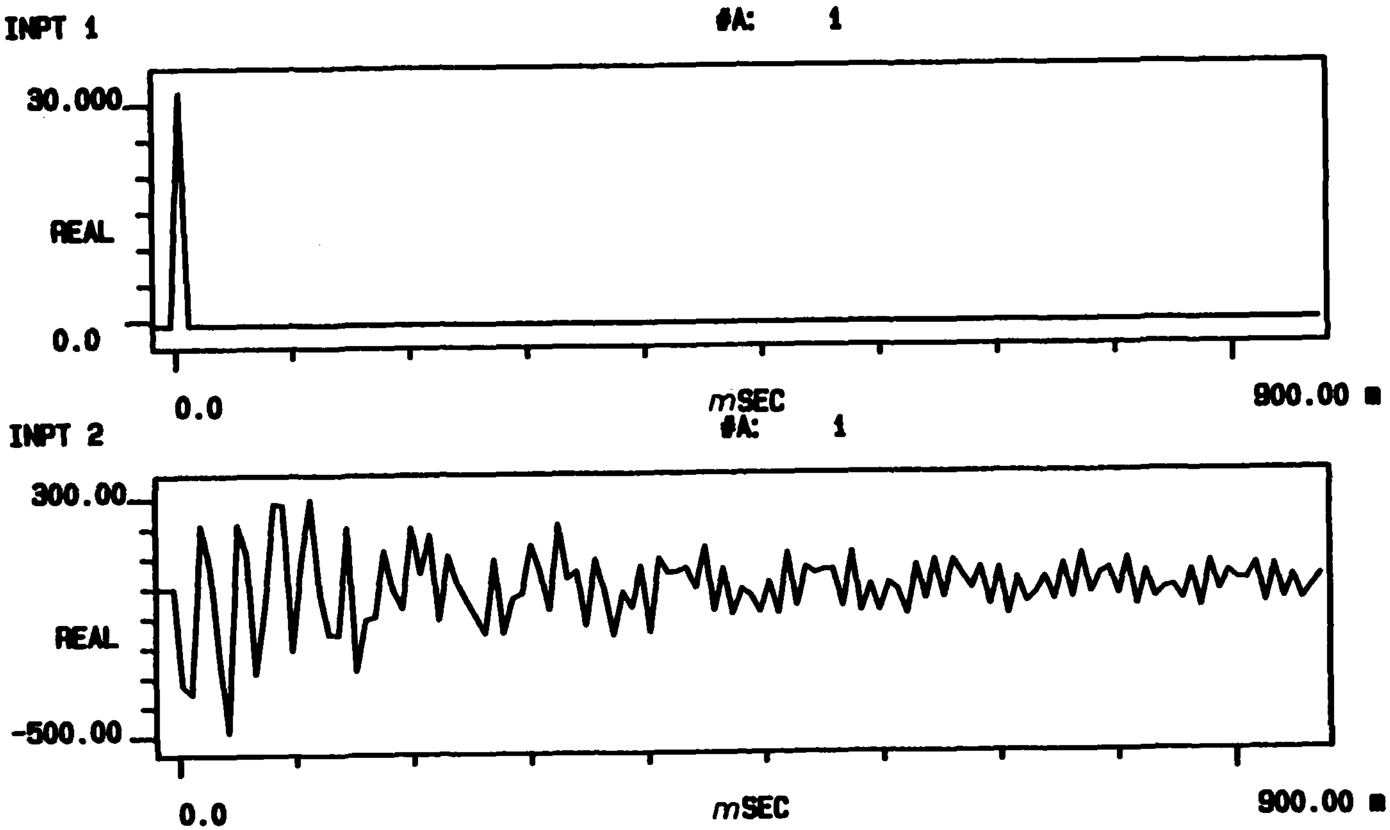

Figure 4. Sample Modal Analyzer Input and Response Signals

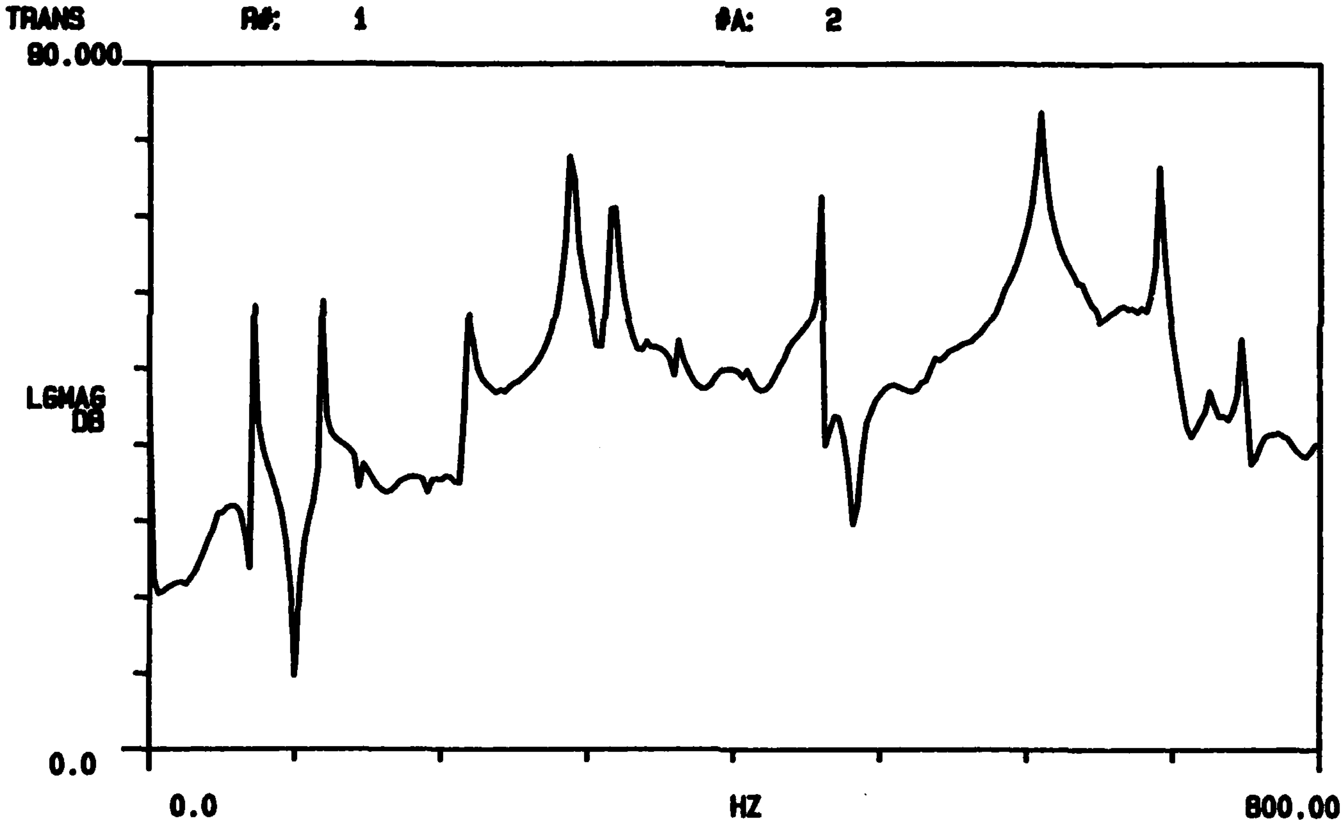

Figure 5. Sample Modal Analyzer Transfer Function

large increase in the value of the function itself. Synonymous with the Laplace transform representation of the transfer function, these poles occur at the natural frequencies of the system being analyzed. Thus, by locating these poles the natural frequencies can be calculated. This transfer function provides one element of the transfer function matrix G. Each element, G_{ij} for example, represents the ratio of the output at point j due to the input at point i. If a structure is defined by N points, a complete transfer function matrix is defined as an N by N matrix. If enough impact data is taken to fill one row or one column of this matrix, the dynamics analyzer can generate the mode shapes and natural frequencies of the structure. (One row or column provides a minimum amount of data, but more can be generated if a greater degree of accuracy is desired.)

The experiment was performed utilizing the shallow cantilevered shell depicted in Figures 6a and 6b. The shell as shown in Fig. 6b was defined to the analyzer to consist of 56 points. These points were chosen to provide a "fine" enough representation of the structure, while at the same time keeping the number of points at which data must be taken to a minimum. The shell dimensions were chosen because some limited numerical results were found for this specific shape (Ref 5). These numerical results were generated using the Ritz method and provide assurance that the experimental setup and methods are accurate.

The shell was rigidly clamped on a fixed support resulting in points 8, 9, 24, 25, 40, 41 and 56 being fixed. After an impact point was chosen, the output response was measured in the radial direction at all 56 points, and in the tangential direction at all but the centerline

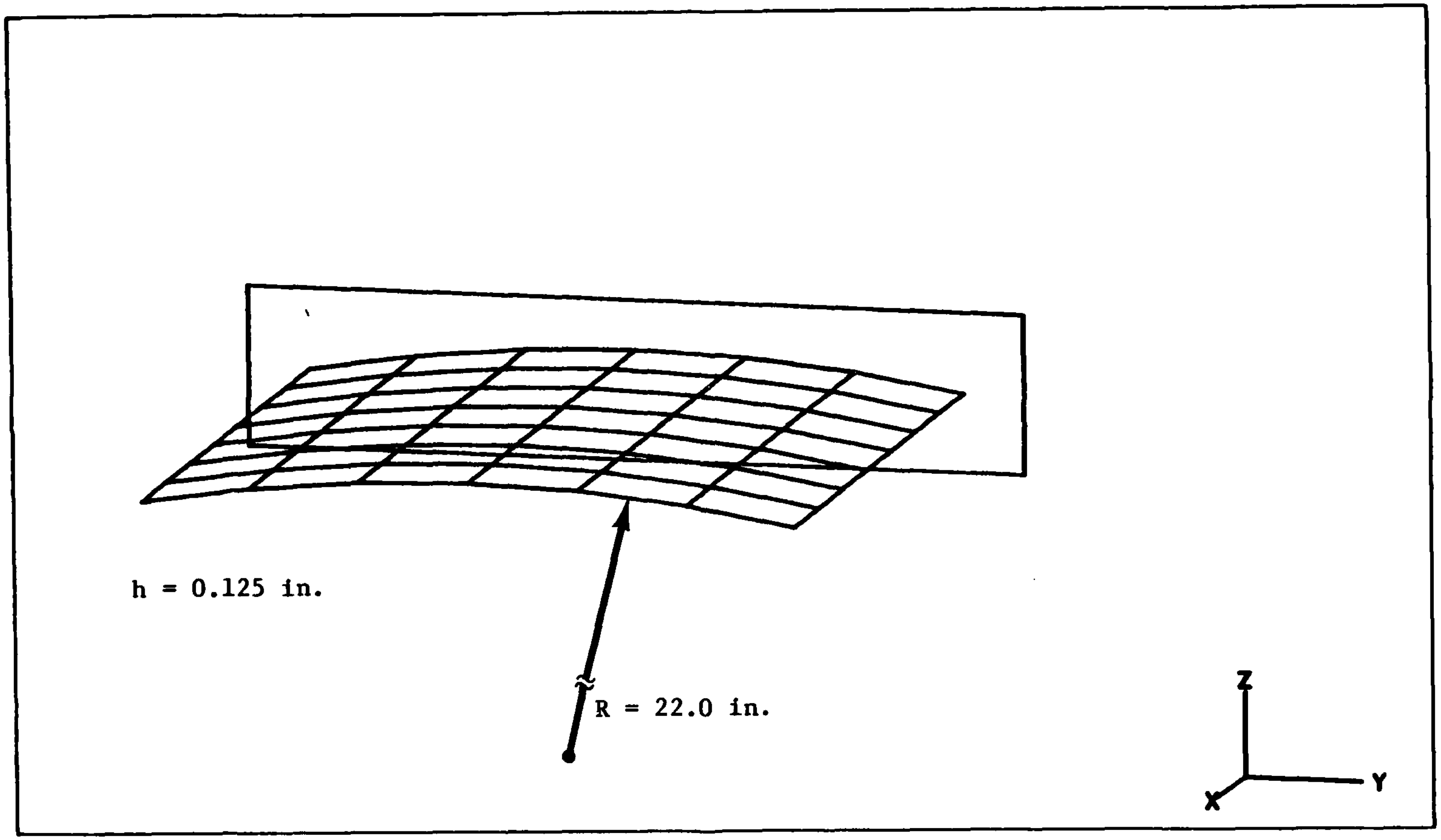

Figure 6a. Orthogonal View of Modal Analyzer Shell

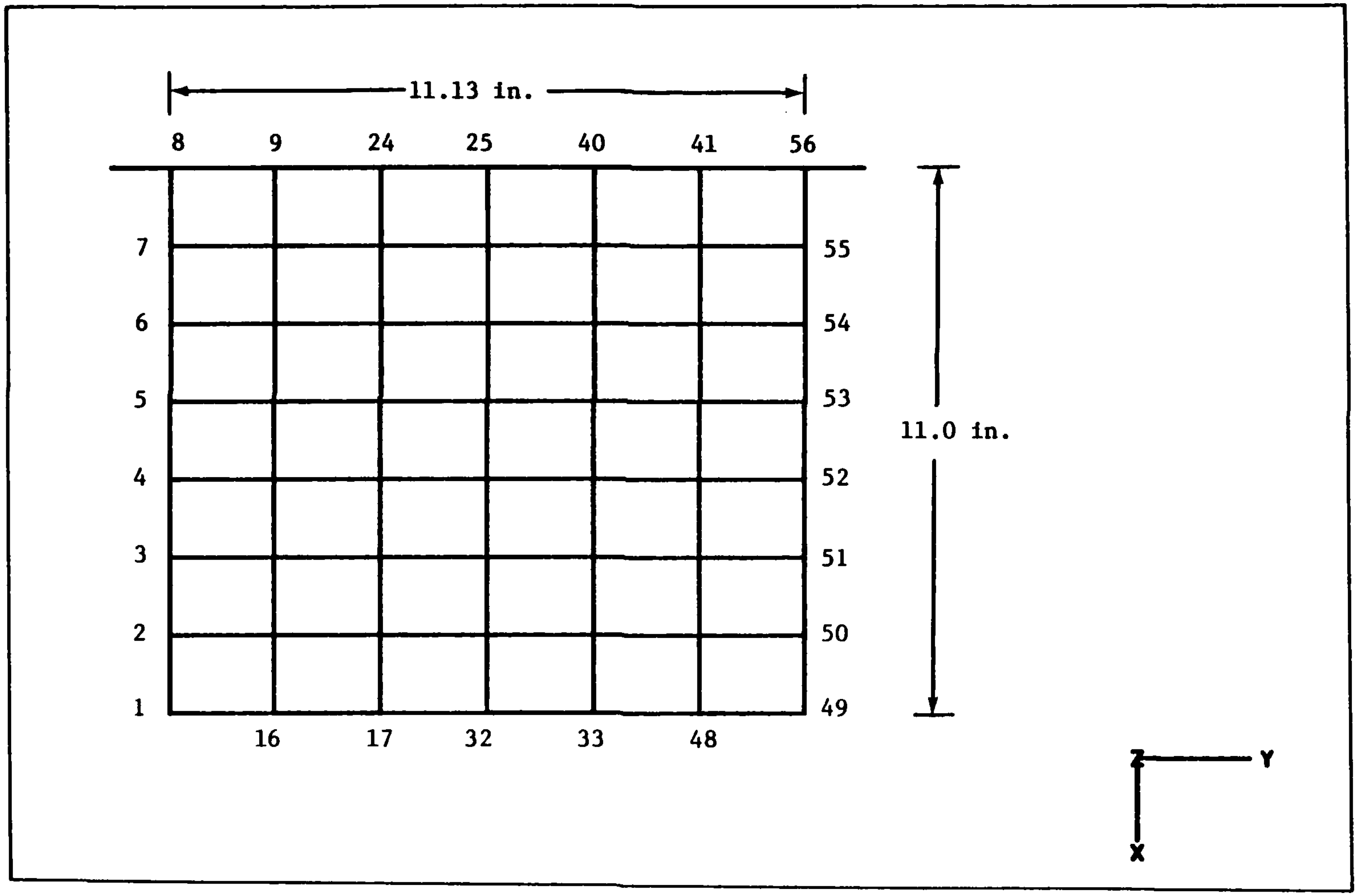

Figure 6b. Top View of Modal Analyzer Shell

and fixed points. The centerline points were omitted because erroneous motion could be generated in the mode shapes if the accelerometer was not placed exactly on the centerline. Having both of these measurements allows the w and v displacements to be included in the mode shapes, thus making them more accurate, but they do not affect the frequency results. As a side note, the accuracy of the results was dependent upon how well the impact point was chosen. If the impact point happened to lie on a nodal line of one of the normal modes, then that mode was not excited sufficiently. Therefore, several trial runs must be performed, or the characteristics of the mode shapes must be known well enough a priori in order to find an acceptable excitation point. For this test, point 6 or 54 was found to work very well.

Experimental Results

Table I compares the experimental results to those calculated using the Ritz method (Ref 5). The results can be seen to compare favorably. The last column of the table shows that the experimental data is between five and ten percent lower than the analytic results. The experimental data is expected to be lower due to the difficulty in achieving a true clamped boundary condition, and the additional mass of the accelerometer. The mode shapes for these natural frequencies (Ref 5) are presented in Fig. 7. From the figure and Table I it can be seen that the first mode of vibration for the shell is a twisting mode, while the second is a bending mode. This is unlike the vibration characteristics of a cantilevered plate, which has a symmetric bending mode for its first fundamental frequency, and a twisting mode for its second (Ref 12). Figures 8 and 9 show the shell's normalized mode shapes as

TABLE I.

Cantilevered Shell Natural Frequencies

Mode No.	Sym-metric	Antisym-metric	Freq (cys/sec) Ritz Method	Experimental	% Diff
1		X	80.38	73.56	8.5
2	X		128.93	121.01	6.1
3	X		232.56	220.80	5.1
4		X	320.78	296.00	7.7
5	X		362.37	325.70	10.1
6		X	497.54	466.24	6.3
7		X	688.70	619.67	10.0
8	X		690.58	--	--

(Accelerometer mass = 4.1 grams = 0.8 % mass of the shell)

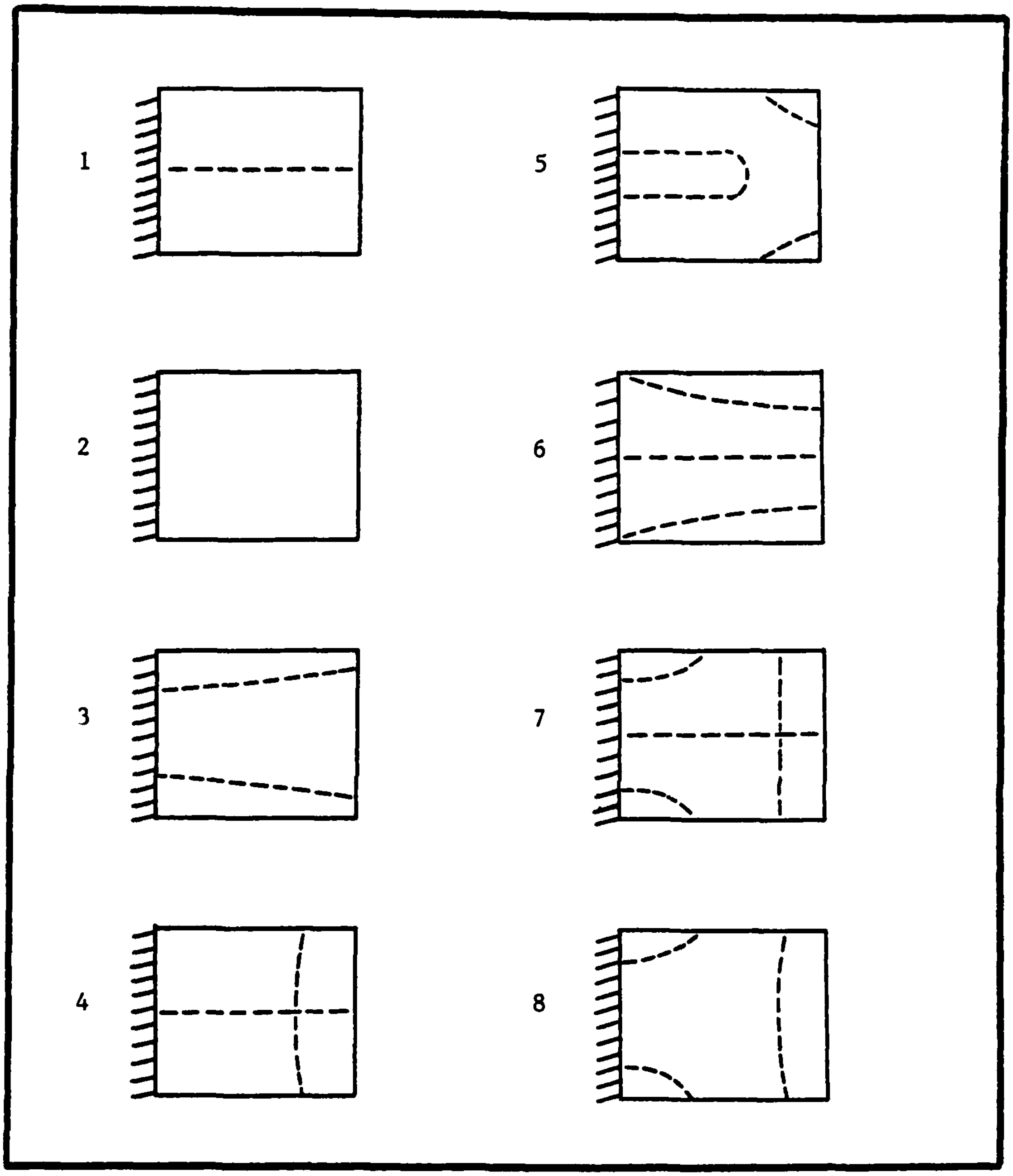

Figure 7. Cantilevered Shell Mode Shapes

calculated by the modal analyzer. Figure 8 clearly shows the antisymmetric twisting motion, while Fig. 9 shows an antisymmetric motion in the circumferential direction. Figure 8 also reveals a reverse bending of the tips of the shell at the two free corners. Since these two areas undergo greater amplitude and hence experience greater accelerations than the other regions of the shell, they are more susceptible to the increased mass of the attached accelerometer. This increase in mass would cause the corners to "lag" behind the normal motion of the shell as shown in the figure. Also, in Fig. 9 a noticeable difference exists between the motion on the left edge of the shell versus the right edge. The left edge happens to be the side where the impact point was chosen (point 6), and therefore should record a slightly greater response to the impact than the far side. This, along with the fact that the motion in Fig. 9 has been magnified several times to make it visible, help to explain this phenomenon, which was also present in the other modes. (See Appendix B for the remaining mode shapes.)

All of the mode shapes calculated by the modal analyzer compared identically to those calculated with the Ritz technique with the exception of the seventh mode. This mode shape, given in Figures 10 and 11, does not agree with the one given in Fig. 7. An investigation into this anomaly revealed that the seventh mode, as determined by the analyzer, is actually a combination of the seventh and eighth modes. Table I shows that the natural frequencies of these two modes are very close together. In addition, Fig. 7 shows that the two mode shapes have very similar nodal lines, but that mode seven is antisymmetric while eight is symmetric. If the two mode shapes are superimposed, the displacements

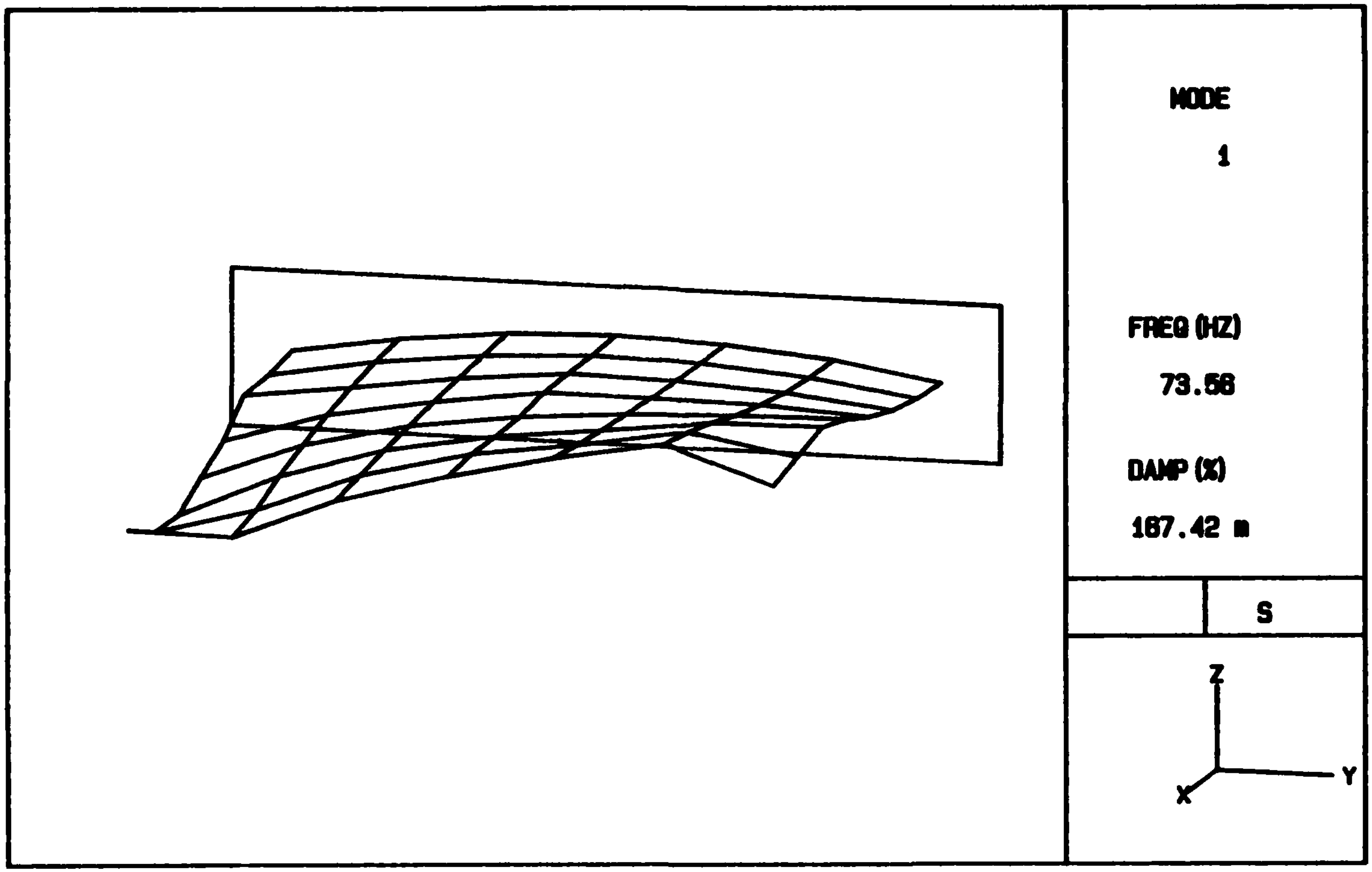

Figure 8. Orthogonal View of Mode 1

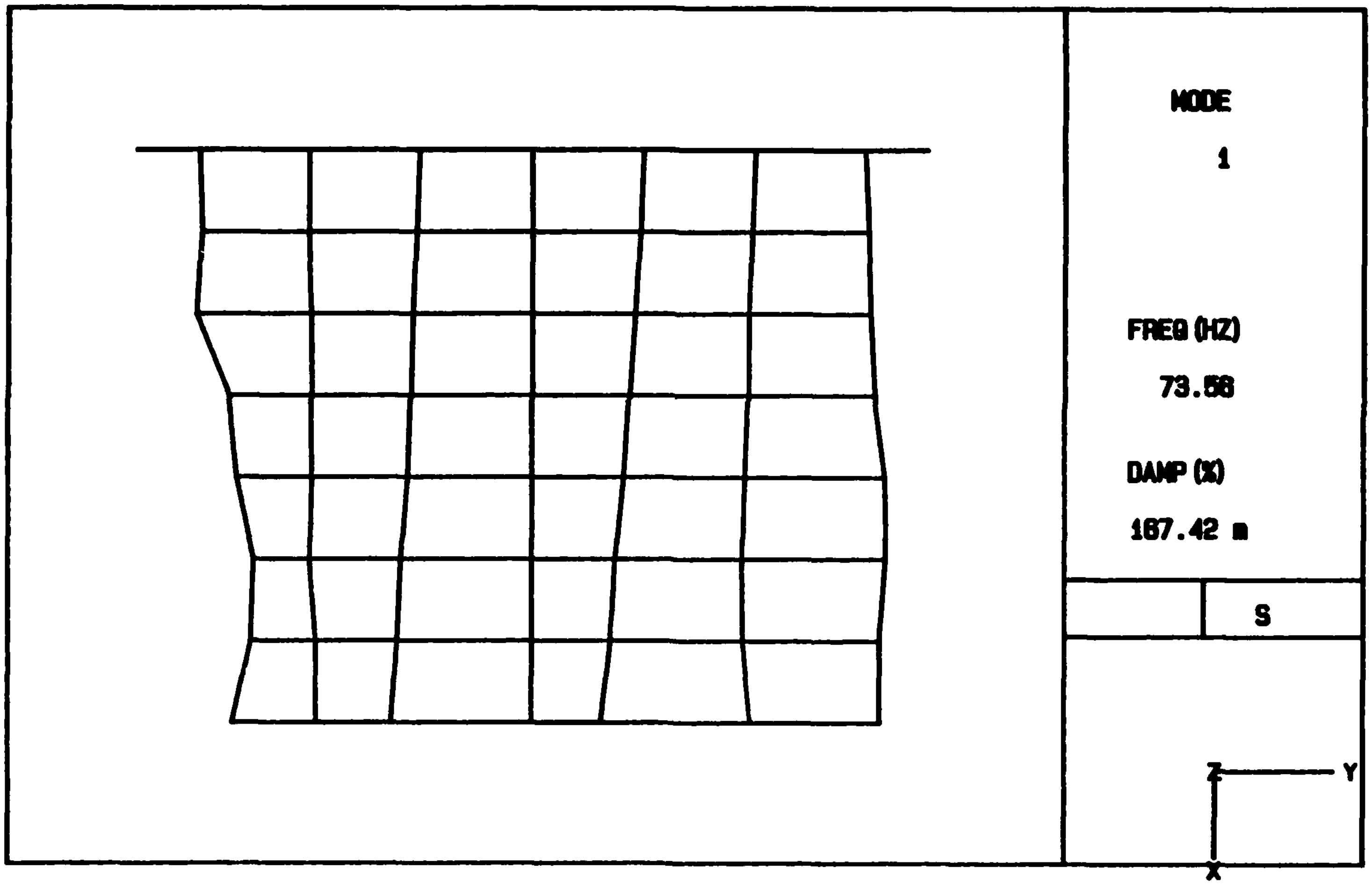

Figure 9. Top View of Mode 1

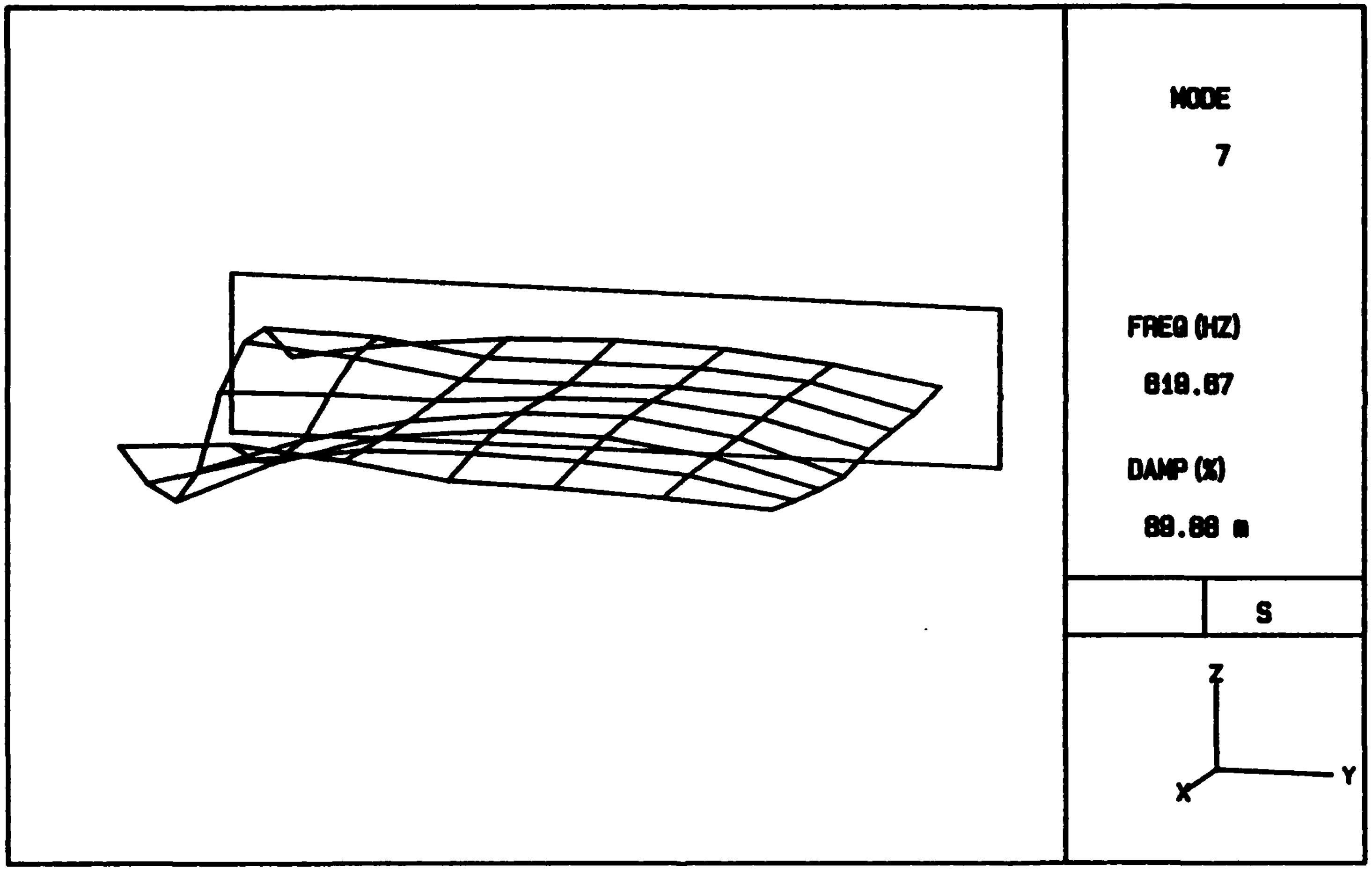

Figure 10. Orthogonal View of Mode 7

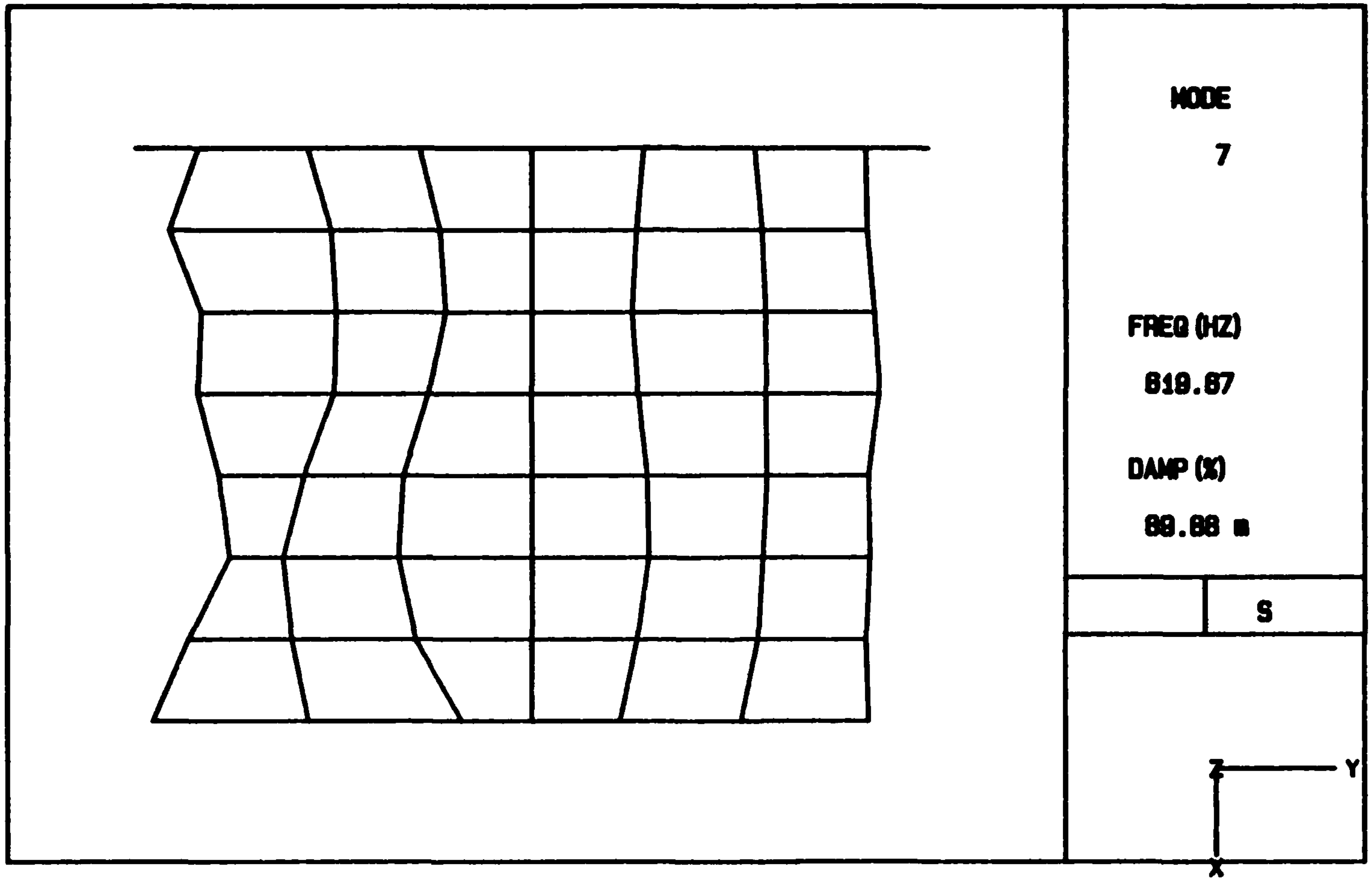

Figure 11. Top View of Mode 7

on one side would cancel each other while on the opposite side they would be additive. This is exactly what is depicted in Fig. 10. Thus, to separate these modes on the analyzer, each one must be individually excited, while the other is suppressed. This could be done by very carefully choosing the impact point, or more easily, by exciting the structure with a shaker at a specific frequency rather than by an impact (see Ref 13).

Computational Method Development

The variational procedure derived earlier and represented by Eq (52), was used as a basis for the development of a computer program to be used to calculate the vibration characteristics of a cantilevered shell. The computer program was written to perform the necessary operations to fill the Q matrix and solve for its determinant. The roots of the equation, and hence the shell's natural frequencies are found in an iterative manner by finding values of ω which cause the determinant to vanish. The computer program consists of a straight-forward coding of Eqs (30), (31), (32) and (47h) to solve for the components ϕ_i, Ψ_i, λ_i and Γ_i. These are then used in conjunction with Eq (52) to fill the Q matrix. All of the required operations were easily performed by internal or system subroutines. A brief description and a listing of the program is given in Appendix C. The program, written in Fortran V, can fill and calculate the determinant of a ten term matrix in approximately one second of computation time on a CDC 6600 digital computer (NOS operating system).

The computer program was developed up to the point of performing all of the necessary calculations to solve for the natural frequencies.

However, the initial program results indicate that further work must be done to improve the accuracy of the program's calculations. Simple tests show that numerical problems are present and are significantly affecting the results. For instance, after solving Eq (32) for its eight roots, one of them must be established as λ_1. While this choice is completely arbitrary at first, the same root must then be chosen each successive iteration to ensure that the determinant of Q behaves as a continuous function. While the choice significantly affects the magnitudes of the Γ_i, and hence the elements of Q itself, it should not change the solutions to Eq (52). Unfortunately, this indeed occurs, and different solutions can be found depending upon how Γ_i is chosen. A way to eliminate this problem must be found before accurate solutions to the problem can be expected.

As mentioned earlier, the matrix Q in Eq (54) es expected to be Hermitian, which would result in the determinant of Q being purely real. Investigating the elements of Q for different values of ω shows that Q is not Hermitian, and that its determinant is complex. However, the matrix diagonal is purely real (another requirement of a Hermitian matrix), and some specific elements are very close to being the complex conjugates of one other. Again, the numerical problems discussed above must be eliminated before any further analysis can be performed in this area.

VII. Conclusions and Recommendations

The variational procedure which has already been used to calculate the vibrational characteristics of a membrane and plate with mixed boundary conditions can be extended to a cylindrical shell. A computational method has been derived to perform all of the necessary operations needed to implement the procedure, but the nature of the calculations makes them sensitive to numerical errors.

The type of problem which may be solved with the variational technique can easily be extended from the cantilevered open cylindrical shell that was presented in this study. Any form of mixed boundary conditions can be implemented by changing the boundary integrals that were evaluated around the four edges. In this manner, virtually any combination of boundary conditions can be investigated. The technique can even be modified to analyze shells with edge cracks. For this case, the boundary integrals must be evaluated along two additional free edges.

In this light, the variational technique is a powerful method which can be applied to problems which cannot be handled by techniques such as the Ritz method and the Galerkin method. Since this flexibility would be applicable to a wide range of practical problems, further work in improving the accuracy of the numerical computations is justified.

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Appendix A

The Betti and Rayleigh Reciprocal Theorem states that if a body is subjected to two systems of body and surface forces, then the work that would be done by the first system T_i, F_i in acting through the displacements u_i' due to the second system of forces is equal to the work that would be done by the second system T_i', F_i' in acting through the displacements u_i due to the first system of forces (Ref 11:391). For a system with no body forces this can be expressed as

$$\int_s T_i u_i' \; ds = \int_s T_i' \; u_i \; ds \tag{A-1}$$

if solutions are assumed to be of the form

$$\begin{aligned} T_i &= \mathrm{Re}(T^n \exp(i\omega t)) \\ u_i &= \mathrm{Re}(u^n \exp(i\omega t)) \\ T_i' &= \mathrm{Re}(T^m \exp(i\omega t)) \\ u_i' &= \mathrm{Re}(u^m \exp(i\omega t)) \end{aligned} \tag{A-2}$$

and are substituted into Eq (A-1), the result is

$$\begin{aligned} &\int_s \mathrm{Re}(T^n \exp(i\omega t)) \quad \mathrm{Re}(u^m \exp(i\omega t)) \quad ds \\ &\quad = \int_s \mathrm{Re}(T^m \exp(i\omega t)) \quad \mathrm{Re}(u^n \exp(i\omega t)) \quad ds \end{aligned} \tag{A-3}$$

This can be written as

$$\int_s \left[\mathrm{Re}(T^n)\cos\omega t - \mathrm{Im}(T^n)\sin\omega t\right]\left[\mathrm{Re}(u^m)\cos\omega t + \mathrm{Im}(u^m)\sin\omega t\right] ds$$

$$= \int_s \left[\mathrm{Re}(T^m)\cos\omega t - \mathrm{Im}(T^m)\sin\omega t\right]\left[\mathrm{Re}(u^n)\cos\omega t - \mathrm{Im}(u^n)\sin\omega t\right] ds \qquad \text{(A-4)}$$

If Hamilton's principle is applied, and these two integrals are integrated with respect to time over one period, the result is

$$\tfrac{1}{2}\int_s \left[\mathrm{Re}(T^n)\mathrm{Re}(u^m) + \mathrm{Im}(T^n)\mathrm{Im}(u^m)\right] ds$$

$$= \tfrac{1}{2}\int_s \left[\mathrm{Re}(T^m)\mathrm{Re}(u^n) + \mathrm{Im}(T^m)\mathrm{Im}(u^n)\right] ds \qquad \text{(A-5)}$$

After multiplying through by two, this can be written as

$$\mathrm{Re}(\bar{T}^n u^m)ds = \mathrm{Re}(T^m \bar{u}^n)ds \qquad \text{(A-6)}$$

for this to be true, then

$$\mathrm{Re}(\bar{T}^n u^m) = \mathrm{Re}(\bar{u}^n T^m) \qquad \text{(A-7)}$$

or

$$-\mathrm{Re}(T^n \bar{u}^m) = -\mathrm{Re}(u^n \bar{T}^m) \qquad \text{(A-8)}$$

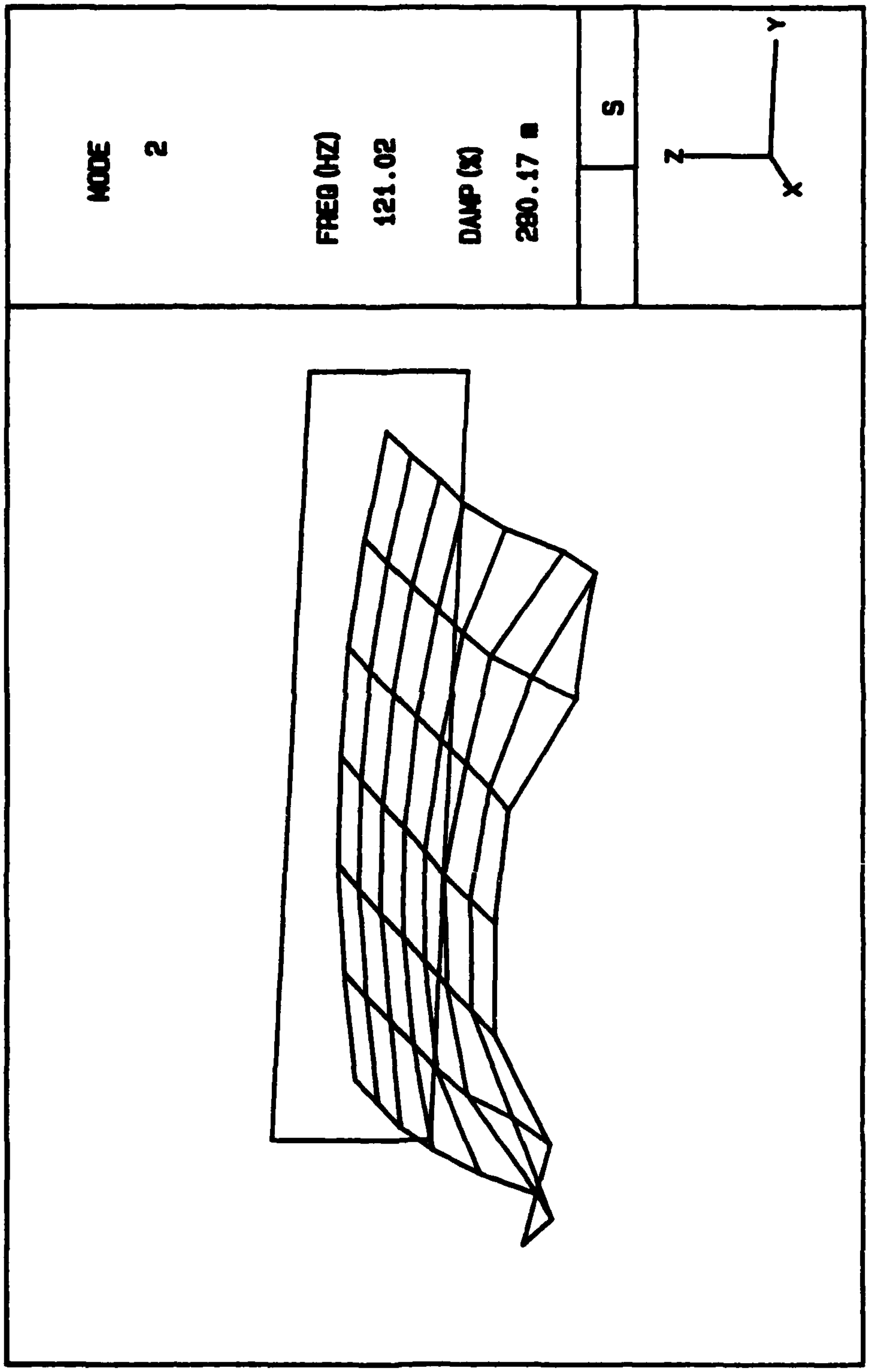
If the matrix element Q_{nm} is defined as $T^n u^m$ then from Eq (A-8)

$$Q_{nm} = \bar{Q}_{mn} \qquad \text{(A-9)}$$

Thus, the matrix Q is Hermitian. Since this matrix is of the same form as that derived in Section V, the Q matrix in Eq (54) should also be Hermitian.

Appendix B

Modal Analyzer Mode Shapes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FREQ (HZ)
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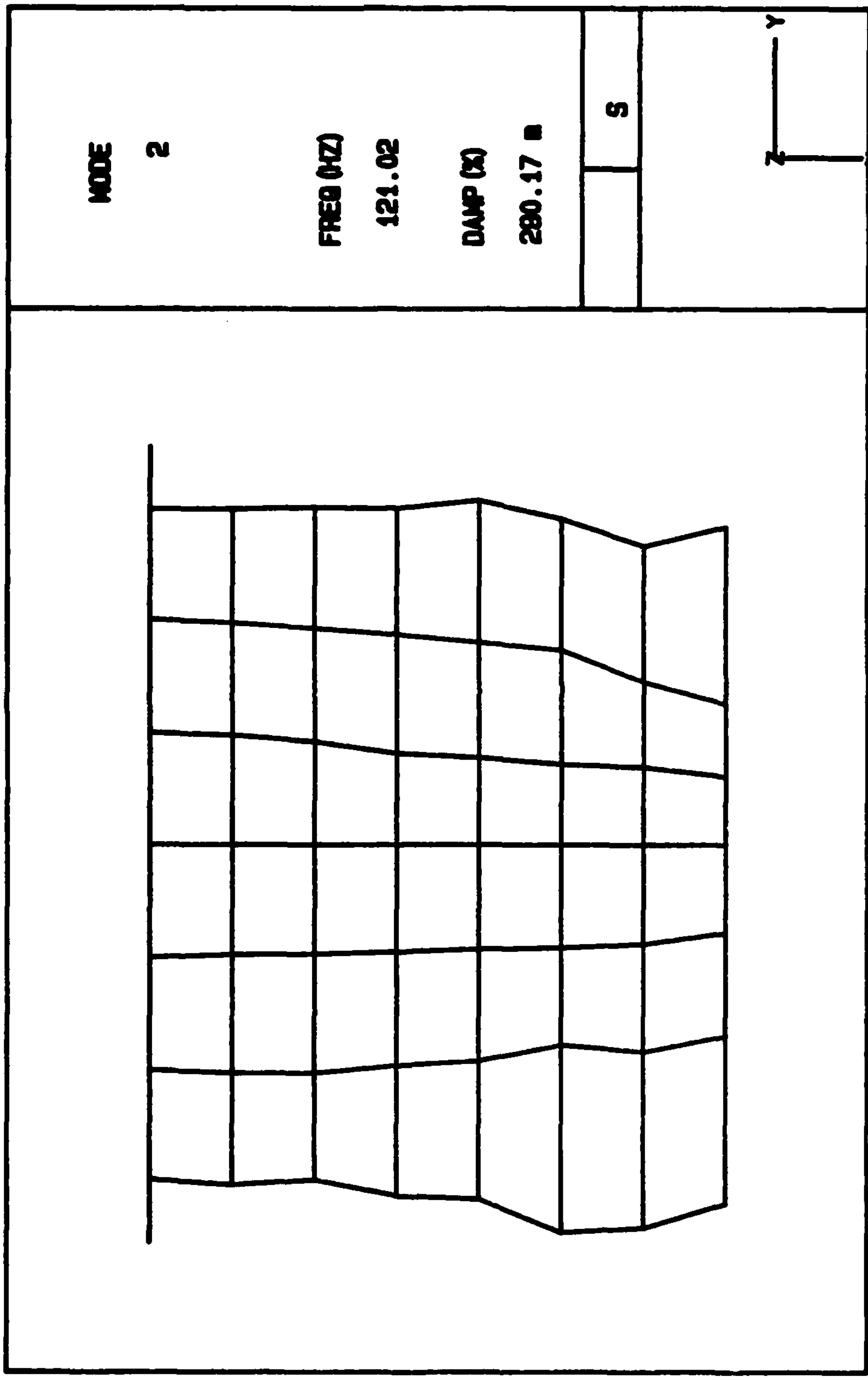
DAMP (%)
290.17
S
Z
Y
X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Z
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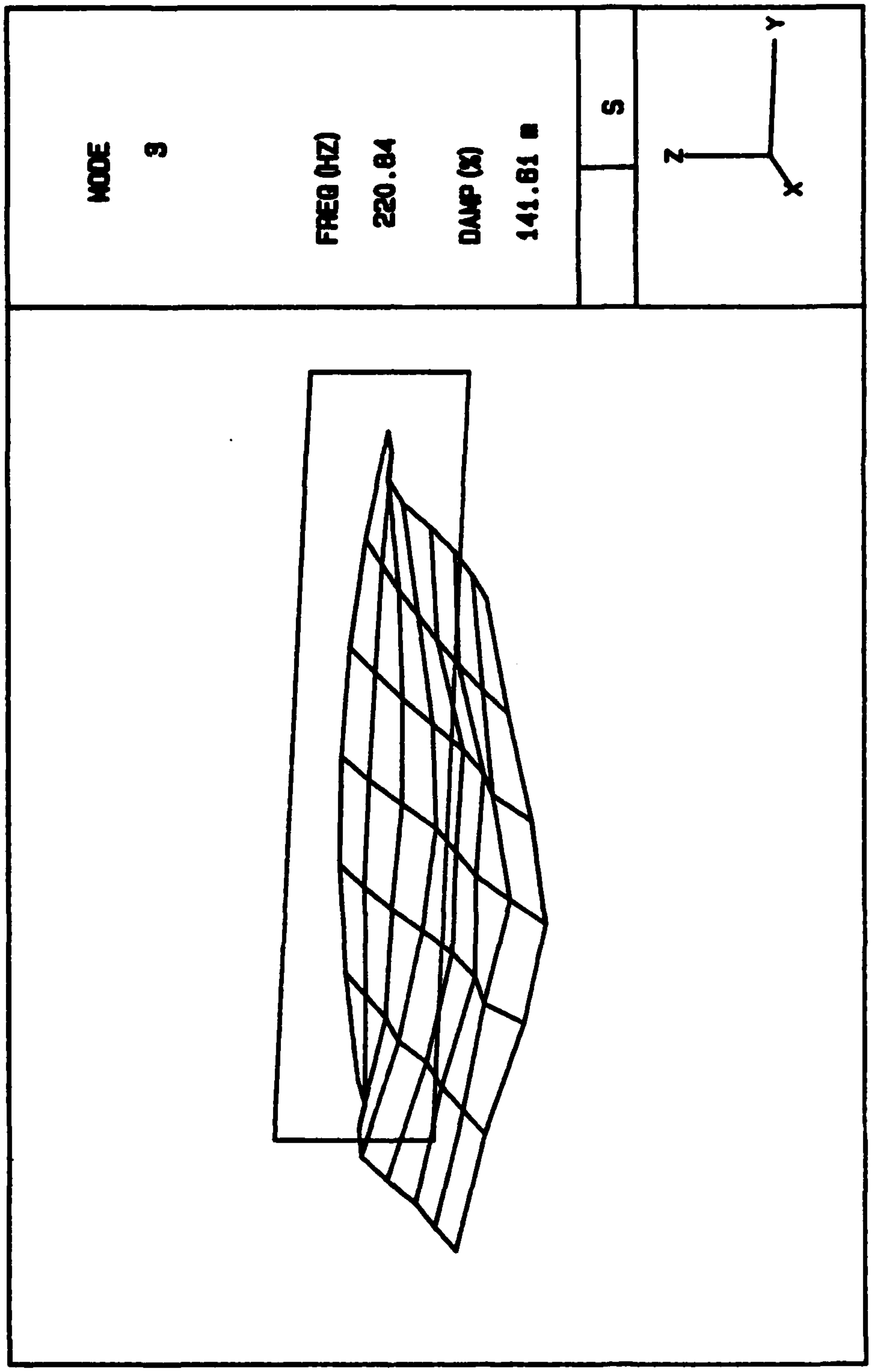
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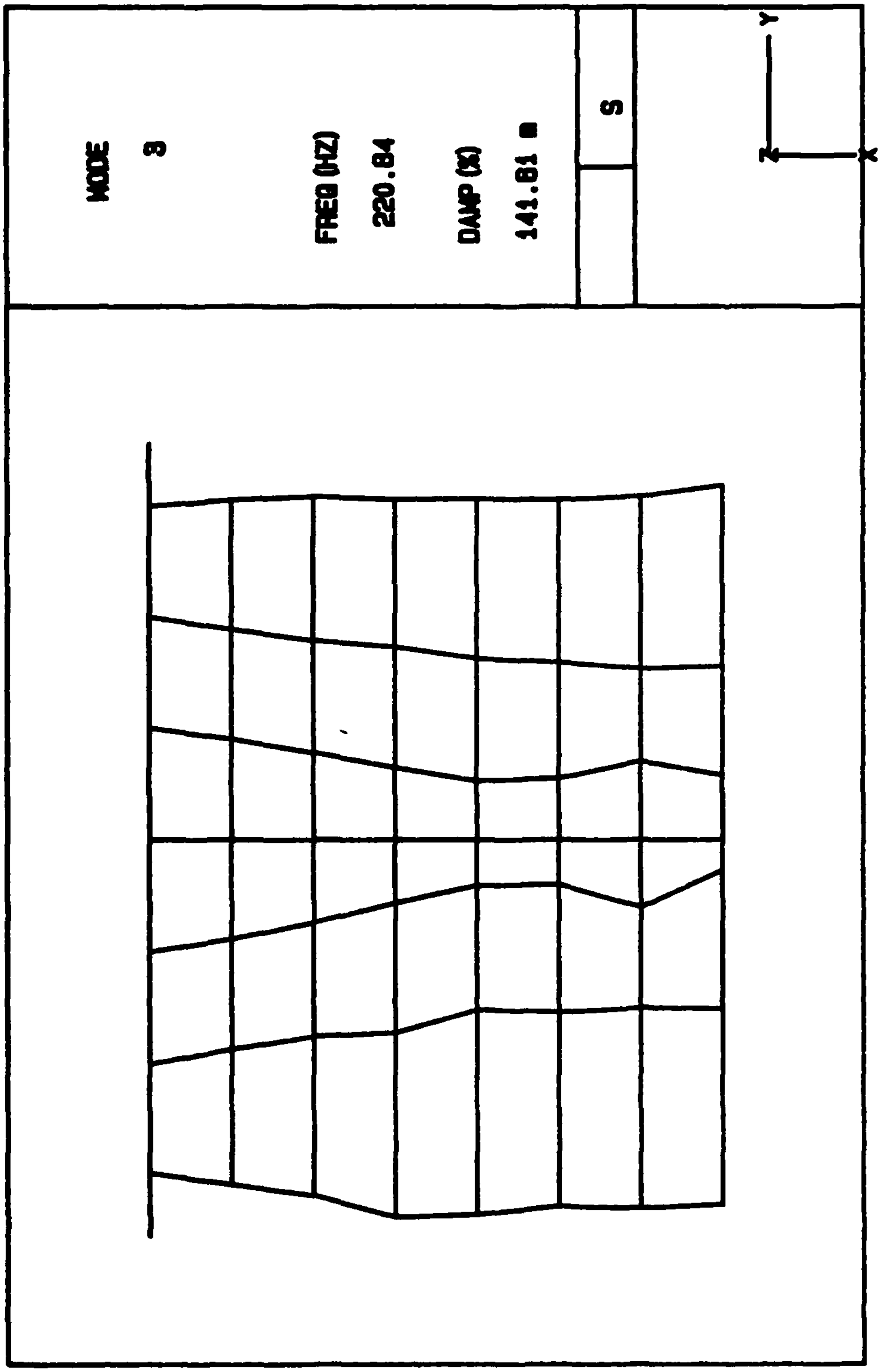
Z
Y
X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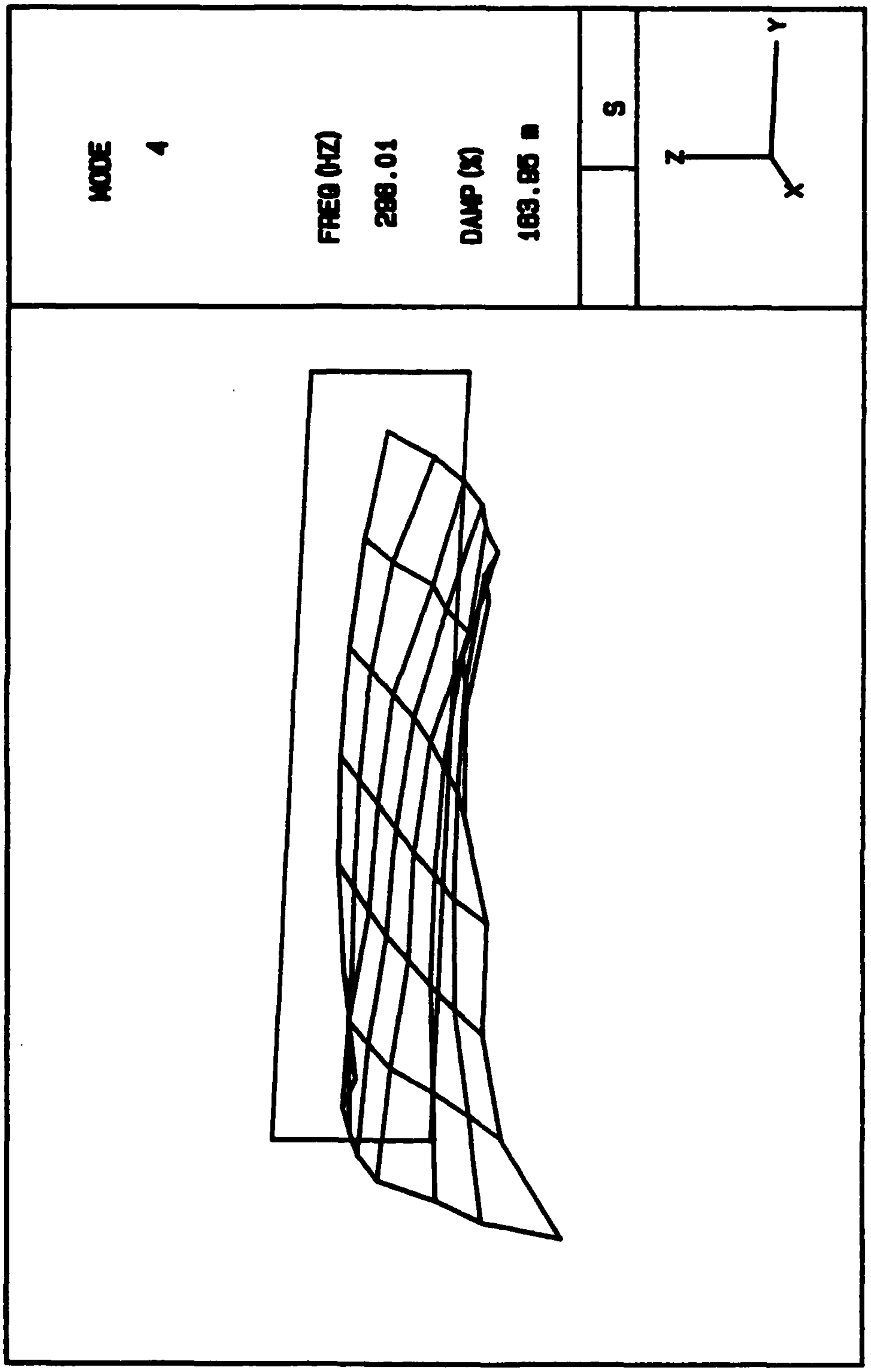
X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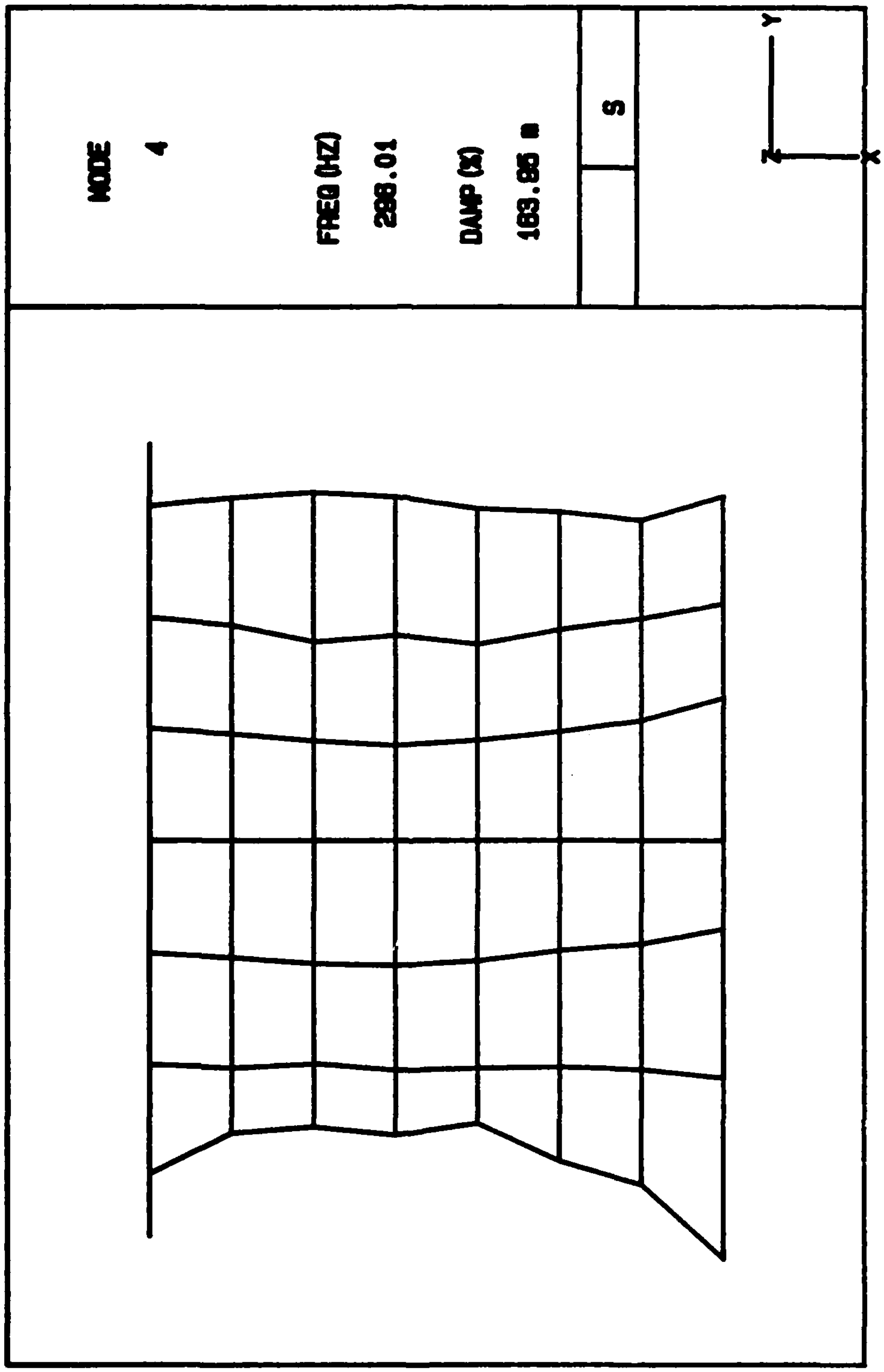
Z
Y
X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DAMP (%)
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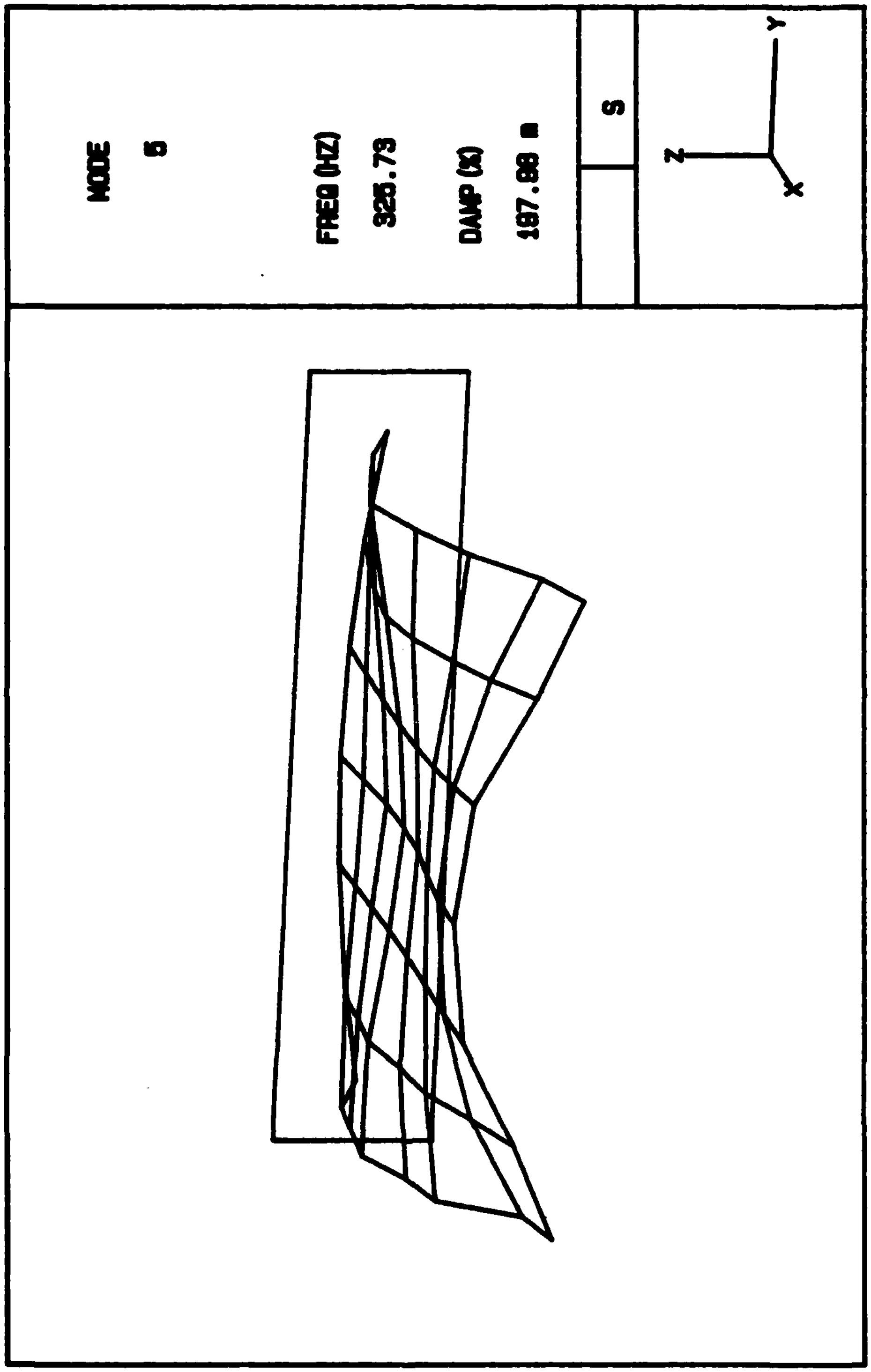
Y
Z
X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DAMP (%)
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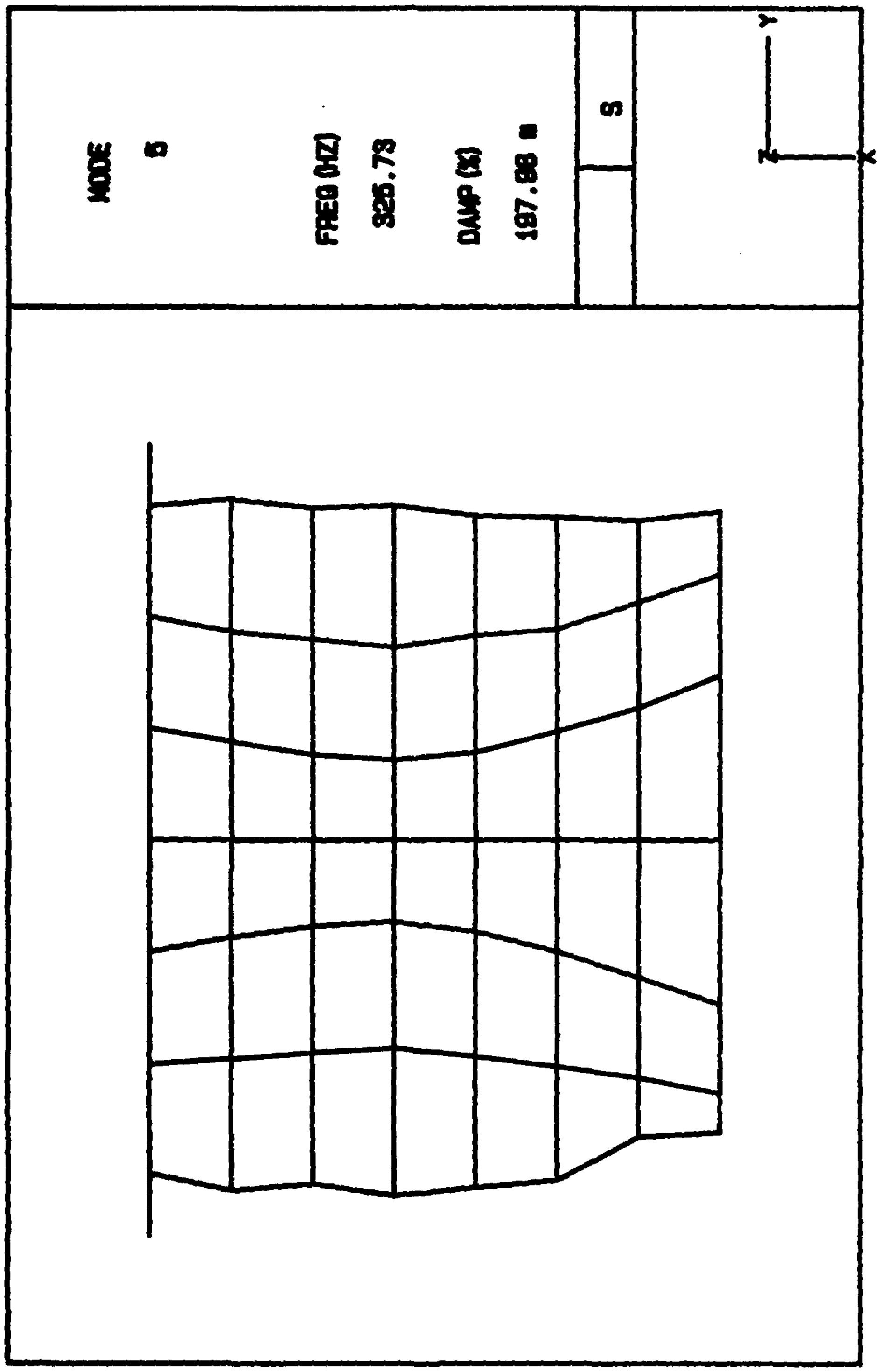
S
Z
Y
X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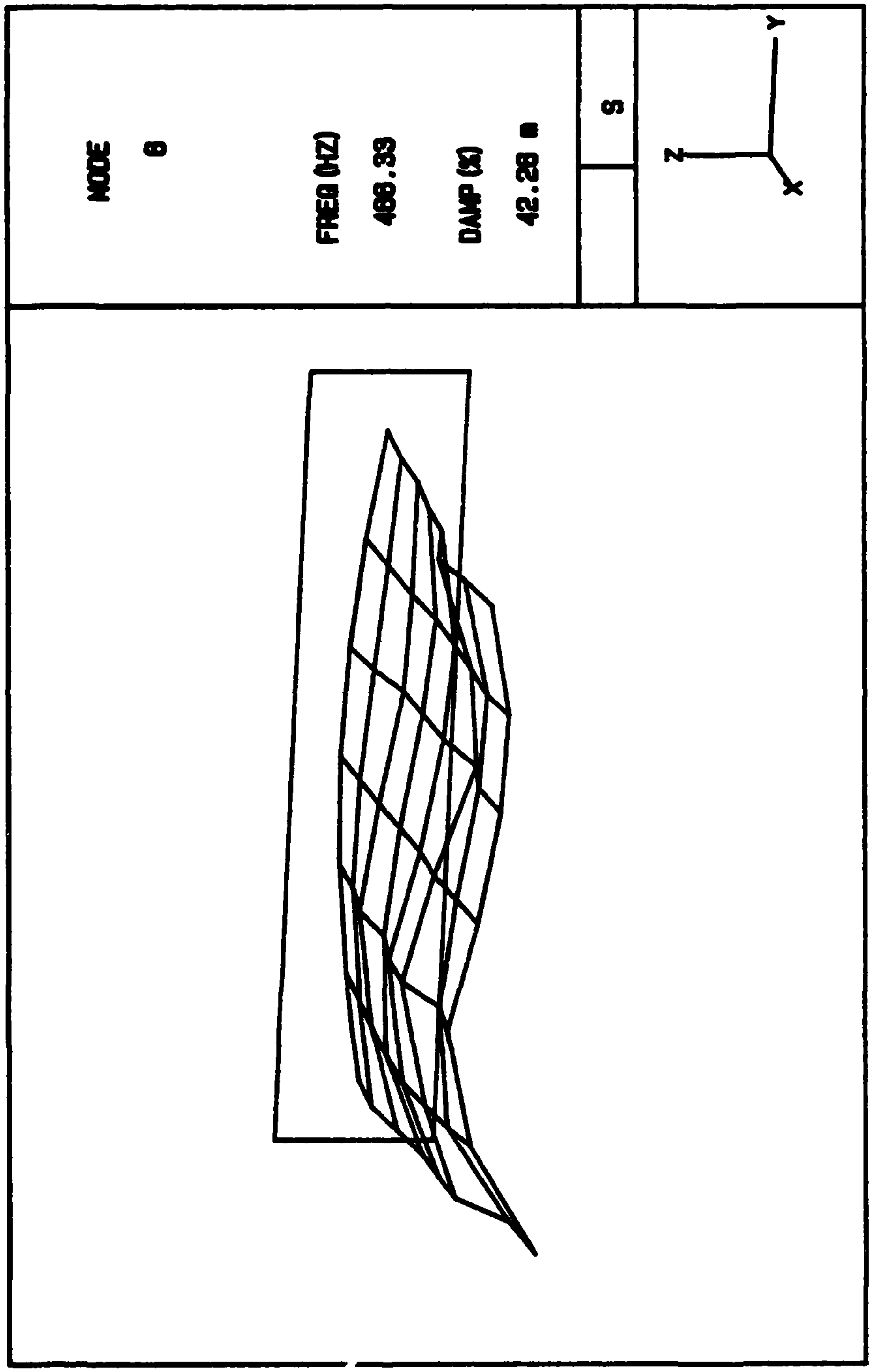
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Z
Y
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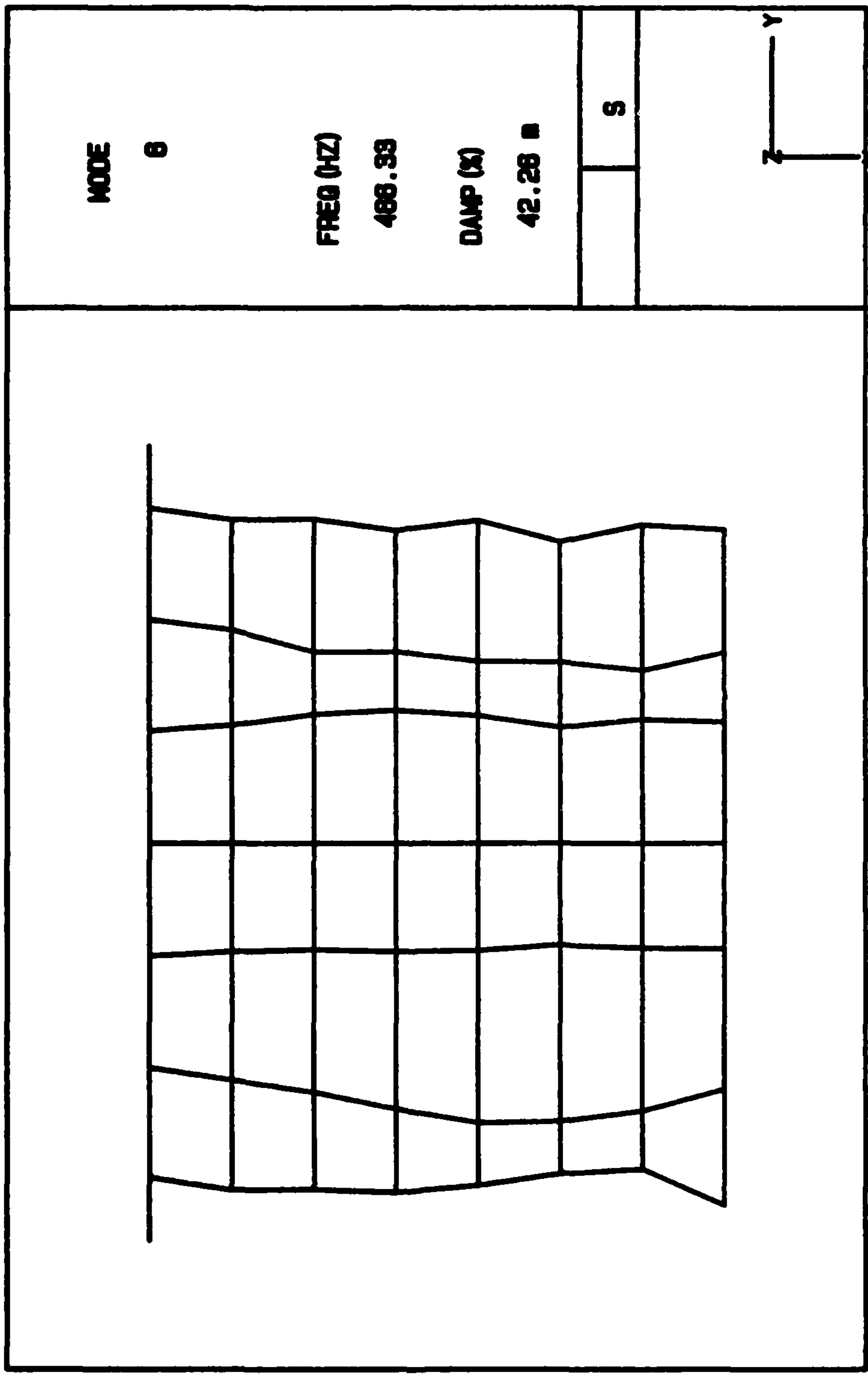
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X

Appendix C

Description of Computer Program

The computer program in this appendix is for the analysis of the vibration characteristics of an open cylindrical cantilevered shell. The program consists of the executive program, Shell, followed by two subroutines, Fill and Poly. The purpose of the program is to calculate the elements of the system matrix, Q, for a given frequency, ω, and then calculate the matrix determinant. The vanishing of the determinant establishes that a natural frequency of the system has been found. The program has not been written to iterate to a solution, but to only sweep across a range of frequencies at a desired step size, while printing out the determinant after each calculation. Since the program can be run interactively, the operator can choose the input parameters to manually iterate as close to a solution as desired. A description and listing of the program and subroutines follow.

Program Shell

Program Shell is the executive routine which reads in the required input parameters, and then after calling the appropriate subroutines, uses a system subroutine to calculate the determinant of Q. The program does this a specified number of times over a range of frequencies.

Subroutine Poly

Subroutine Poly calculates the values of λ_i, Ψ_i, ϕ_i and Γ_i and stores them in array C. It performs these calculations for $n = 0$ to NTERM-1. It also fills the array CJ with the complex conjugate of the

values in array C. This subroutine utilizes two system routines. One is needed to solve an eighth order equation for its roots, and the other to solve a system of complex equations for their unknown coefficients.

Subroutine Fill

Subroutine Fill uses the C and CJ arrays to calculate the elements of the Q matrix. The coding of the program is representative of Eqs (52) and (54) in the text.

```
      PROGRAM SHELL
      COMPLEX C,CJ,Q
      COMMON/KK/TD(18)
      COMMON/KB/ C(8,4,0:29),CJ(8,4,0:29),Q(30,30)
      COMPLEX B(30),DETC
      DIMENSION WC(30)
      EQUIVALENCE(TD(8),D1),(TD(9),D2)
      EQUIVALENCE(TD(12),V1),(TD(13),V2),(TD(14),V3)
      EQUIVALENCE(TD(6),V),(TD(7),W),(TD(15),MAG)
      EQUIVALENCE(TD(16),D3),(TD(18),NTERM)
      TD(17)=99.
      PRINT*,'INPUT NUMBER OF TERMS- '
      READ(5,*)NTERM
      PRINT*,'INPUT W1 AND NUMBER OF ITERATIONS'
      READ(5,*)W1,N2
      PRINT*,'INPUT INC'
      READ(5,*)YINC
      TD(1)=22.
      TD(3)=.095
      TD(10)=.506
      TD(11)=11.
      TD(2)=.00026
      TD(5)=10400000.
      TD(6)=.3
      PRINT*,' '
      PRINT*,'INPUT MAG FACTOR'
      READ(5,*)MAG
      V1=1-V
      V2=1.-V**2
      V3=3.-V
      D1=TD(5)*TD(3)**3/(12*(1-TD(6)**2))
      D2=TD(5)*TD(3)/(2+2*TD(6))
```

```
      D3=TD(5)*TD(3)/V2
      W=W1-YINC
      DO 5 N=1,N2
      W=W+YINC
      CALL POLY
      CALL FILL
      DO 51 I=1,10
   51 WRITE(6,50)(Q(I,J),J=1,10)
   50 FORMAT(/,1X,2(5('(',1P2E10.3,')',1X),/,1X))
      CALL LEQT1C(Q,NTERM,30,B,1,30,1,WC,IER)
      DETC=(1.0,0.0)
      DO 6 I=1,NTERM
      IPVT=WC(I)
      IF(IPVT.NE.I)DETC=-DETC
      DETC=DETC*Q(I,I)
    6 CONTINUE
      PRINT *,' ',N,'  W=',W,'  DETC= ',DETC
    5 CONTINUE
      END
```

```
      SUBROUTINE FILL
      COMPLEX C,CJ,Q
      COMMON/KK/TD(18)
      COMMON/KB/ C(8,4,0:29),CJ(8,4,0:29),Q(30,30)
      EQUIVALENCE(TD(10),TH),(TD(6),V),(TD(8),D)
      EQUIVALENCE(TD(9),D2),(TD(13),V2),(TD(12),V1)
      EQUIVALENCE(TD(5),E),(TD(3),H),(TD(15),MAG)
      EQUIVALENCE(TD(1),R),(TD(16),D1),(TD(18),NTERM)
      COMPLEX T(7),DFF
      REAL INT,L
      ACC=0.000001
      L=TD(11)
      IM=7
      DO 100 M=0,NTERM-1
      IF(M.NE.0)IM=8
      M1=M+1
      DO 100 N=0,NTERM-1
      IF(N.NE.0)IN=8
      IF(N.EQ.0)IN=7
      N1=N+1
      DO 2 I=1,7
    2 T(I)=(0.0,0.0)
      SC=SIN(N*TH/2)*COS(M*TH/2)
      CS=COS(N*TH/2)*SIN(M*TH/2)
      IF(N.EQ.0.OR.M.EQ.0) THEN
      INT=0.0
      ELSE
      IF(N.NE.M)THEN
      INT=SIN((M-N)*TH/2)/(M-N)-SIN((M+N)*TH/2)/(M+N)
      ELSE
      INT=TH/2-SIN(N*TH)/(2*N)
      END IF
      END IF
```

```
      DO 10 I=1,IN
      DO 10 J=1,IM
      T(6)=T(6)+(D2*CJ(J,4,M)*(CJ(J,3,M)*CJ(J,1,M)/L)
     >)*C(I,4,N)*C(I,3,N)
      T(1)=(N*C(I,1,N)/(R*L))*C(I,4,N)*CJ(J,4,M)*
     >EXP(C(I,1,N)+CJ(J,1,M))
      DFF=C(I,1,N)+CJ(J,1,M)
      IF(ABS(REAL(DFF)).LE.ACC.AND.ABS(AIMAG(DFF)).LE.ACC)THEN
      T(7)=L*C(I,4,N)*CJ(J,4,M)
      ELSE
      T(7)=L*(C(I,4,N)*CJ(J,4,M)/(C(I,1,N)+CJ(J,1,M)))*
     >(EXP(C(I,1,N)+CJ(J,1,M))-1)
      END IF
      T(2)=T(2)+(C(I,3,N)*C(I,1,N)/L-C(I,2,N)*N/R)*CJ(J,2,M)*T(7)
      T(3)=T(3)+(C(I,3,N)*N/R+1/R+V*C(I,2,N)*C(I,1,N)/L)*
     >CJ(J,3,M)*T(7)
      T(4)=T(4)+((2-V)*(C(I,1,N)/L)**2-(N/R)**2)*T(7)
      T(5)=T(5)+((N/R)**2-V*(C(I,1,N)/L)**2)*(M+CJ(J,3,M))*T(7)
   10 CONTINUE
      Q(M1,N1)=2*D2*SC*T(2)+2*D1*CS*T(3)-4*D*V1*SC*T(1)+
     >2*D*(N/R)*SC*T(4)+2*D*CS*(1/R)*T(5)+T(6)*INT*R
      Q(M1,N1)=Q(M1,N1)/(10**MAG)
      Q(M1,N1)=Q(M1,N1)/(10**MAG)
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE POLY
      COMPLEX C,CJ,Q
      COMMON/KK/TD(18)
      COMMON/KB/ C(8,4,0:29),CJ(8,4,0:29),Q(30,30)
      REAL J(6)
      DIMENSION A(9)
      REAL L
      COMPLEX Z(8),GR(7,7),B(7),T,DENOM,SWP,DFF,OLD1,OLD2
      COMPLEX WA(63),WK(7)
      EQUIVALENCE(TD(1),R),(TD(2),P),(TD(3),H),(TD(5),E)
      EQUIVALENCE(TD(6),V),(TD(7),W),(TD(8),D1),(TD(9),D2)
      EQUIVALENCE(TD(11),L),(TD(12),V1),(TD(13),V2),(TD(14),V3)
      EQUIVALENCE(TD(18),NTERM)
      DO 50 N=0,NTERM-1
      DO 5 I=1,9
5     A(I)=0.0
      CI=(12.*R*R)/(H*H)
      D=P*R*R*V2*W*W/E
      IF(N.NE.0)THEN
      G=(R/(N*L))**2
      J(1)=V1*N**8/CI
      J(2)=D*V3*N**6/CI
      J(3)=D*D*N**4/CI
      J(4)=D*V1*N**4
      J(5)=D*V3*N**2
      J(6)=J(5)*D
      A(1)=-J(1)*G**4
      A(3)=(4*J(1)-J(2))*G**3
      A(5)=(-6*J(1)+3*J(2)-2*J(3)+J(4))*G**2-(R/L)**4*V1*V2
      A(7)=(4*J(1)-3*J(2)-2*J(4)+4*J(3)+2*D*((N*V)**2)-J(5)+
     >J(6))*G
      A(9)=2*D**3-2*D*D-J(6)-2*J(3)+J(5)-D*2*N*N+J(4)+J(2)-J(1)
      CALL ZPOLR(A,8,Z,IER)
```

```
      IF(N.EQ.1.OR.N.EQ.5)THEN
      ELSE
      END IF
      ELSE
      G=(R/L)
      A(1)=G**7/CI
      A(3)=(G**5)*D/(CI*V)
      A(5)=(1/V-D/R-V)*G**3
      A(7)=-(D-D*D)*G
      IER=0
      CALL ZPOLR(A,6,Z,IER)
      Z(7)=(0.,0.)
      Z(8)=(0.,0.)
      END IF
      IF(TD(17).GT.5.AND.N.EQ.0)OLD2=Z(1)
      IF(N.EQ.0)OLD1=OLD2
      XMX=10**5
      IFLG=0
      DO 10 I=1,8
      C(I,1,N)=Z(I)
      DFF=OLD1-C(I,1,N)
      XM=SQRT((REAL(DFF))**2+(AIMAG(DFF))**2)
      IF(XM.EQ.XMX)PRINT*,'OH-OH'
      IF(XM.LT.XMX)THEN
      IFLG=I
      XMX=XM
      ELSE
      END IF
   10 CONTINUE
      IF(IFLG.NE.1)THEN
C     PRINT*,'*** SWAP A ROOT ***'
      SWP=C(1,1,N)
      C(1,1,N)=C(IFLG,1,N)
```

```
      C(IFLG,1,N)=SWP
      Z(1)=C(1,1,N)
      Z(IFLG)=C(IFLG,1,N)
      ELSE
      END IF
      OLD1=C(1,1,N)
      DO 20 I=1,8
      IF(N.NE.0)THEN
      T=1-G*Z(I)**2
      DENOM=(2/V1)*D*D-(V3/V1)*D*N*N*T+N**4*T*T
      C(I,2,N)=(-Z(I)*R/L)*(2*V*D/V1+N*N*(1+V*G*Z(I)**2))/DENOM
      C(I,3,N)=-N*(-2*D/V1+N*N*(1-(2+V)*4*G*Z(I)**2))/DENOM
      ELSE
      C(I,2,N)=(-V*Z(I)/(R*L))/((Z(I)/L)**2+D/(R*R))
      C(I,3,N)=(0.,0.)
      END IF
   20 CONTINUE
      DO 30 I=2,8
      GR(1,I-1)=C(I,2,N)
      GR(2,I-1)=(1.,0.)
      GR(3,I-1)=Z(I)
      GR(4,I-1)=(V*N*N/(R*R)-Z(I)**2/(L*L))*EXP(Z(I))
      GR(5,I-1)=(C(I,2,N)*Z(I)/L+V*(C(I,3,N)*N/R+1/R))*EXP(Z(I))
      GR(6,I-1)=((Z(I)/L)**2+((N/R)**2)*(V-2))*Z(I)*EXP(Z(I))
      GR(7,I-1)=(D2*(C(I,3,N)*Z(I)/L-C(I,2,N)*N/R)+(D1/(R*R))*V1*
     >(N*Z(I)/L))*EXP(Z(I))
   30 CONTINUE
      B(1)=-C(1,2,N)
      B(2)=-(1.,0.)
      B(3)=-Z(1)
      B(4)=-(V*N*N/(R*R)-Z(1)**2/(L*L))*EXP(Z(1))
      B(5)=-(C(1,2,N)*Z(1)/L+V*(C(1,3,N)*N/R+1/R))*EXP(Z(1))
      B(6)=-((Z(1)/L)**2+((N/R)**2)*(V-2))*Z(1)*EXP(Z(1))
```

```
      B(7)=-(D2*(C(1,3,N)*Z(1)/L-C(1,2,N)*N/R)+(D1/(R*R))*V1*
     >(N*Z(1)/L))*EXP(Z(1))
      IF(N.NE.0) CALL LEQ2C(GR,7,7,B,1,7,0,WA,WK,IER)
      IF(N.EQ.0) CALL LEQ2C(GR,6,7,B,1,7,0,WA,WK,IER)
      IF(N.EQ.0)THEN
      B(7)=(0.,0.)
      ELSE
      END IF
      DO 40 I=2,8
   40 C(I,4,N)=B(I-1)
      C(1,4,N)=(1.,0.)
   50 CONTINUE
      OLD2=C(1,1,1)
      TD(17)=0.
      DO 60 I=1,8
      DO 60 LL=1,4
      DO 60 K=0,NTERM-1
   60 CJ(I,LL,K)=CMPLX(REAL(C(I,LL,K)),-AIMAG(C(I,LL,K)))
      RETURN
      END
```

Vita

Jeffrey V. Kouri was born on 25 December 1955 in Munising, Michigan. Upon graduation from the United States Air Force Academy in May 1978, he received the degree of Bachelor of Science in Aeronautical Engineering and a commission in the United States Air Force. He served as an aircraft store performance engineer at the Air Force Armament Laboratory, Eglin AFB, Fort Walton Beach, Florida until entering The Air Force Institute of Technology in June 1982.

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This report develops a variational technique for the analysis of the vibration characteristics of an open cylindrical cantilevered shell. The technique is developed by modifying Reissner's principle, which normally applies to static problems, through the use of Hamilton's principle so that it applies to dynamic problems. The variational technique is first derived in general for an elastic system, and then specifically tailored to an open cylindrical cantilevered shell. The technique is implemented by first finding a general solution which satisfies the equations of motion for a cylindrical shell. A method is then formulated to use this general solution to construct a set of trial solution functions. With the variational method, the coefficients to this trial solution function are then calculated so that the function not only satisfies the equations of motion, but also the boundary conditions around the four edges of the shell. A computer method was developed to perform the necessary calculations to implement the variational procedure, but preliminary results have shown that numerical problems must be eliminated before accurate results can be expected.

Experimental data for an open cylindrical cantilevered shell was also collected on a modal analyzer. The results are presented and discussed.

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